POSTECH 이성익 교수의

양자 세계에 관한 강연

- 15장 -

편집 도우미: POSTECH 학부생 임향택

Chapter 15

Time-Dependent Perturbation Theory

$$H = H_0 + V(t)$$

assumption

$$H_0 \psi_n = E_n \psi_n$$

풀어야 할 방정식

$$H\psi = i\hbar \frac{\partial \psi}{\partial t}$$

$$H(t) = \sum_n C_n(t)\psi_n(t)$$

$$= \sum_n C_n(t)\psi_n \cdot e^{-i\frac{E_n}{\hbar}t}$$

위의 equation에 넣는다.

$$i\hbar \frac{\partial}{\partial t}\sum_n C_n(t)\psi_n \cdot e^{-i\frac{E_n}{\hbar}t} = H\sum_n C_n(t)\psi_n e^{-i\frac{E_n}{\hbar}t}$$

$$= \left[H_0 + V(t)\right]\sum_n C_n(t)\psi_n \cdot e^{-i\frac{E_n}{\hbar}t}$$

$$= \sum_n C_n(t)\cdot \psi_n E_n e^{-i\frac{E_n}{\hbar}t} + \sum_n V(t) C_n(t)\psi_n e^{-i\frac{E_n}{\hbar}t}$$

$$\text{左} = \sum_n i\hbar \frac{\partial C_n(t)}{\partial t}\psi_n e^{-i\frac{E_n}{\hbar}t} + \sum_n C_n(t)\psi_n E_n e^{-i\frac{E_n}{\hbar}t}$$

$\int \psi_m^* \cdot dt$ 곱

$$i\hbar \frac{\partial C_m(t)}{\partial t} e^{-i\frac{E_m}{\hbar}t} = \sum_n \langle \psi_m | V(t) | \psi_n \rangle C_n(t) e^{-i\frac{E_n}{\hbar}t}$$

$$i\hbar \frac{\partial C_m(t)}{\partial t} = \sum_n \langle \psi_m | V(t) | \psi_n \rangle C_n(t) e^{-i\left(\frac{E_n - E_m}{\hbar}\right)t}$$

$$\therefore i\hbar \frac{\partial C_m(t)}{\partial t} = \sum_n C_n(t)\cdot \langle \psi_m | V(t) | \psi_n \rangle e^{i\left(\frac{E_m - E_n}{\hbar}\right)t}$$

Initial Condition

$$C_n(0) = \delta_{nx}$$

$$\therefore i\hbar \frac{dC_m}{dt} = V_{mI}(t) e^{i\frac{(E_m - E_I)}{\hbar}t}$$

The solution of these equations,

$$C_I(t) = 1 + \frac{1}{i\hbar}\int_0^t V_{II}(t)\,dt$$

$$C_n(t) = \frac{1}{i\hbar}\int_0^t V_{nI}(t) e^{\frac{i}{\hbar}(E_n - E_I)t}\,dt$$

Constant Perturbation

$$C_I(t) = 1 + \frac{V_{II}}{i\hbar}t$$

$$C_n(t) = \frac{V_{nI}}{E_n - E_I}\cdot\frac{1}{i\hbar}\cdot\left(1 - e^{\frac{i}{\hbar}(E_n - E_I)t}\right) \qquad (n \neq I)$$

Initial state가 어떻게 변화하나?

$$= C_I(t)\psi_I e^{-i\frac{E_I}{\hbar}t}$$

$$= \psi_I e^{-i\frac{E_I}{\hbar}t}\left(1 - \frac{i}{\hbar}V_{II}t\right)$$

$$= \psi_I e^{-i\frac{E_I}{\hbar}t}\cdot e^{-i\frac{V_{II}}{\hbar}t}$$

$$= \psi_I e^{-\frac{i}{\hbar}(E_I + V_{II})t}$$

phase만 변한다.

$$|C_n(t)|^2 = \frac{|V_{nI}|^2}{(E_n - E_I)^2}\left|1 - e^{\frac{i}{\hbar}(E_n - E_I)t}\right|^2$$

$$= \frac{|V_{nI}|^2}{|E_n - E_I|^2}\left|2 \cdot \sin\frac{E_n - E_I}{\hbar}t\right|^2$$

이 식은 $C_n(t)$가 매우 작을 때 성립한다.

1. 만약 V_{nI}가 작다면 O.K.

2. 만약 $E_n \neq E_I$ 매우 다르면

$$|C_n|^2_{\max} = \frac{4|V_{nI}|^2}{|E_n - E_I|^2} \rightarrow \text{zero}$$

Transition을 보자.

$$\int |C_n(t)|^2 dE_n = 4|V_{nI}|^2 \int_{-\infty}^{\infty} \sin^2\frac{(E_n - E_I)t}{2\hbar} \cdot \frac{dE_n}{(E_n - E_I)^2}$$

$$= 4|V_{nI}|^2 \int_{-\infty}^{\infty} \sin^2 x \cdot \left(\frac{t}{2\hbar x}\right)^2 \cdot dx \frac{2\hbar}{t} \qquad \left[\frac{(E_n - E_I)}{2\hbar}t = x\right]$$

$$= 4|V_{nI}|^2 \cdot \frac{t^2}{4\hbar^2}\frac{2\hbar}{t}\int_{-\infty}^{\infty}\frac{\sin^2 x}{x^2}dx$$

$$= \frac{2t}{\hbar}|V_{nI}|^2 \cdot \pi \qquad \left(\because \int_{-\infty}^{\infty}\frac{\sin^2 x}{x^2}dx = \pi\right)$$

Hence

$$|C_n(t)|^2 = \frac{2\pi}{\hbar}|V_{nI}|^2 \cdot t \cdot \delta(E_n - E_I)$$

Total transition probability

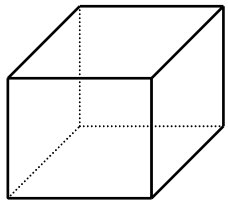
$$\int |C_n(t)|^2 \rho(E_n) dE_n = \frac{2\pi t}{\hbar}\int |V_{nI}|^2 \cdot \delta(E_n - E_F)\rho(E_n) dE_n$$

$$= \frac{2\pi t}{\hbar}|V_{nI}|^2 \rho(E_F)$$

This quantity increase linearly with t

$$W_{FI} = \frac{2\pi}{\hbar}|V_{FI}|^2 \rho(E_F)$$

Box normalization



$$H_0 = \frac{p^2}{2m}$$

$$\psi_I = \frac{1}{\sqrt{L^3}} \exp\left(i\vec{k}\cdot\vec{r}\right), \quad \psi_F = \frac{1}{\sqrt{L^3}} \exp\left(i\vec{k}'\cdot\vec{r}\right)$$

1. Normalization Condition이 만족된다.
2. Periodic Boundary Condition

$$KL = 2n\pi$$

$$\therefore k_x = \frac{2n_x\pi}{L}, k_y = \frac{2n_y\pi}{L}, k_z = \frac{2n_z\pi}{L}$$

$$n = \left(n_x^2 + n_y^2 + n_z^2\right)^{\frac{1}{2}}$$

$$\rho(E)dE = n^2 dn d\Omega$$

$$= \left(\frac{L}{2\pi}\right)^3 k^2 dk d\Omega$$

$$n = \frac{kL}{2\pi}$$

$$E = \frac{p^2}{2m} = \frac{\hbar^2 k^2}{2m}$$

$$dE = \frac{\hbar^2}{\cancel{2}m} \cancel{2} k dk$$

$$= \frac{V}{8\pi^3} k \cdot kdk \cdot d\Omega$$

$$= \frac{V}{8\pi^3} \cdot \frac{mdE}{\hbar^2} \cdot \sqrt{\frac{2mE}{\hbar^2}} d\Omega$$

$$\therefore \rho(E) = \frac{V}{8\pi^3} \cdot \frac{m}{\hbar^2} \sqrt{\frac{2mE}{\hbar^2}} d\Omega$$

$$= \frac{V}{8\pi^3} \cdot \frac{m}{h^3} \sqrt{2mE} d\Omega$$

$$V_{FI} = \left(\psi_{F,} V \psi_I\right)$$

$$= \frac{1}{L^3} \int \exp\left(-i\vec{k}'\cdot\vec{r}\right) V(r) \exp\left(i\vec{k}\cdot\vec{r}\right) d\vec{r}$$

$$W_{FI} = \frac{2\pi}{\hbar}\left|V_{FI}\right|^2 \rho\left(E_F\right)$$

$$\therefore \quad = \frac{2\pi}{\hbar}\left|V_{FI}\right|^2 \cdot \frac{V}{8\pi^3} \cdot \frac{m}{\hbar^3} \cdot \sqrt{2mE}\, d\Omega$$

$$= \frac{2\pi}{\hbar} \cdot \frac{1}{V} \cdot \frac{1}{8\pi^3} \cdot \frac{m}{\hbar^3} \sqrt{2mE}\, d\Omega \cdot \left|\int \exp\left(-i\vec{k}' \cdot \vec{r}\right) V\left(r\right) \exp\left(i\vec{k} \cdot \vec{r}\right) d\vec{r}\right|^2$$

$$W_{FI} = S_i \sigma\left(\vec{k}, \vec{k}'\right) d\Omega$$

$$= \frac{\hbar k}{mL^3} \sigma\left(\vec{k}, \vec{k}'\right) d\Omega$$

Combine this expression

$$\sigma\left(\vec{k}, \vec{k}'\right) = \left|\frac{1}{4\pi} \frac{2m}{\hbar^2} \int V\left(r\right) \exp\left[i\left(\vec{k} - \vec{k}'\right) \cdot \vec{r}\right] d\vec{r}\right|^2$$

Schrödinger Equation

$$Eu\left(\vec{r}\right) = \left[-\frac{\hbar^2}{2\mu}\nabla^2 + V\left(r\right)\right] u\left(\vec{r}\right)$$

$$E = \frac{\hbar^2 k^2}{2\mu}$$

$$\rightarrow \therefore \left[\nabla^2 + k^2 - \mathrm{V}\left(r\right)\right] u\left(\vec{r}\right) = 0 \qquad \mathrm{V}\left(r\right) = \frac{2\mu}{\hbar^2} V\left(r\right)$$

Let $u\left(\vec{r}\right) = \sum_{l,m} R_l\left(r\right) Y_{lm}\left(\theta, \phi\right)$

with $R_l\left(r\right)$ satisfying

$$0 = \frac{1}{r^2} \frac{d}{dr}\left(r^2 \frac{dR_l}{dr}\right) + \left(k^2 - \frac{l\left(l+1\right)}{r^2} - \mathrm{V}\left(r\right)\right) R_l$$

or $\chi_l\left(r\right) \equiv rR_l\left(r\right)$, then

$$\chi_l'' + \left[k^2 - \frac{l\left(l+1\right)}{r^2} - \mathrm{V}\left(r\right)\right] \chi_l\left(r\right) = 0$$

만약 $r \rightarrow$ 大

$$\chi_l'' + k^2 \chi_l = 0$$

$$\therefore \chi_l = e^{\pm ikr}$$

Put $\chi_l \equiv v_l(r) e^{\pm ikr}$

$$\Rightarrow v_l'' \pm 2ikv_l' - k^2 v_l + \left(k^2 - \frac{l(l+1)}{r^2} - \mathrm{V}(r) \right) v_l = 0$$

for smooth Potential, $|v_l''| << |kv_l'|$

$$\pm 2i \cdot \frac{v_l'}{v_l} = \mathrm{V}(r) + \frac{l(l+1)}{r^2}$$

$$\therefore \pm 2i \ln v_l = \int \left[\mathrm{V}(r) + \frac{l(l+1)}{r^2} \right] dr + \text{const}$$

If $\mathrm{V}(\mathrm{r}) \underset{r\to\infty}{\rightarrow} \frac{\text{const}}{r^{1+s}} \qquad (S >> 0)$

$$\pm 2i \ln v_l \underset{r\to\infty}{\rightarrow} \quad \text{const}$$

$$\underset{\text{Coulomb potential}}{\rightarrow} \ln r \qquad (S = 0)$$

If $S > 0$

$$\chi_l(r) \rightarrow A \sin(kr + \delta) \text{ and } B \cos(kr + \delta)$$

Thm.

For $S > 0$, $\exists a > 0$, such that for $r > a$,

$$V_{\text{eff}} = 0$$

Exact Solution for $r > a$

$$\frac{d^2 R_l}{dr^2} + \frac{2}{r}\frac{dR_l}{dr} + \left(k^2 - \frac{l(l+1)}{r^2}\right)R_l = 0$$

$$\rightarrow \begin{aligned} R_l &= A_l^{'} j_l(kr) + B_l^{'} n_l(kr) \\ &= A_l\left(j_l(kr)\cos\delta_l - n_l(kr)\sin\delta_l\right) \end{aligned}$$

$r \rightarrow \infty$

$$R_l \rightarrow A_l\left[\frac{\cos\left(kr - \frac{l+1}{2}\pi\right)}{kr}\cos\delta_l - \frac{\sin\left(kr - \frac{l+1}{2}\pi\right)}{kr}\sin\delta_l\right]$$

$$= \frac{A_l}{kr}\sin\left(kr - \frac{l\pi}{2} + \delta_l\right)$$

δ_l: phase shift

Born Approximation

Consider $(H - E)\psi(\vec{r}) = \chi(\vec{r})$

a) $H = H(r)$ Hermitian Operator

b) Assume known $\left(H(r) - E\right)\phi_E(\vec{r}) = 0$

$\{\phi_E(\vec{r})\}$ is complete set

Let $\psi(\vec{r}) = \int dE^{'} A(E^{'})\phi_{E^{'}}(\vec{r})$

then

$$\begin{aligned} \chi(r) &= (H - E)\psi(\vec{r}) \\ &= (H - E)\int dE^{'} A(E^{'})\phi_{E^{'}}(\vec{r}) \\ &= \int dE^{'} A(E^{'})(E^{'} - E)\phi_{E^{'}}(\vec{r}) \end{aligned}$$

Multiply $\int d^3r\phi^*_{E''}(\vec{r})$

$$\int d^3r\phi^*_{E''}(\vec{r}) = \int dE' A(E')(E'-E)\underline{\int d^3r\phi_{E'}(\vec{r})\phi^*_{E''}(\vec{r})}$$
$$= A(E'')(E''-E) \qquad \delta(E'-E'')$$

$$\therefore A(E') = \frac{1}{E'-E}\int d^3r'\phi^*_{E'}(\vec{r}')\cdot\chi(\vec{r}')$$

$$\psi(\vec{r}) = \int dE'\phi_{E'}(\vec{r})\cdot\frac{1}{E'-E}\int d^3r'\phi^*_{E'}(\vec{r}')\chi(\vec{r}')$$

$$\therefore \quad = \int d^3r'\left[\int dE'\frac{\phi_{E'}(\vec{r})\phi^*_{E'}(\vec{r}')}{E'-E}\right]\chi(\vec{r})$$

Let $G^{(\pm)}_E(\vec{r},\vec{r}') = \lim_{\varepsilon\to 0}\int dE'\frac{\phi_{E'}(\vec{r})\phi^*_{E'}(\vec{r}')}{E'-(E\pm i\varepsilon)}$

then $\psi(\vec{r}) = \int d^3r' G^{(\pm)}_E(\vec{r},\vec{r}')\chi(\vec{r}')$

$$(H-E)G^{(\pm)}_E(\vec{r},\vec{r}') = \delta(\vec{r}-\vec{r}')$$
$$(H-E)\phi = 0$$

$$\therefore \psi(\vec{r}) = \phi + \int d^3r' G^{(\pm)}_E(\vec{r},\vec{r}')\chi(\vec{r}')$$

where $(H-E)\phi = 0$

$$(H-E)G^{(\pm)}_E(\vec{r},\vec{r}') = \delta(\vec{r}-\vec{r}')$$

For Schrödinger equation

$$\left(-\frac{\hbar^2}{2\mu}\nabla^2 - E\right)\psi(\vec{r}) = -V(\vec{r})\psi(\vec{r})$$

$$\left(-\frac{\hbar^2}{2\mu}\nabla^2 - E\right)\phi(\vec{r}) = 0 \qquad \therefore \phi(\vec{r}) = Ae^{i\vec{k}\cdot\vec{r}} + Be^{-i\vec{k}\cdot\vec{r}}$$

$$\phi_E(\vec{r}) = \frac{1}{(2\pi)^{\frac{3}{2}}}e^{i\vec{k}\cdot\vec{r}}$$
$$= \phi_k(\vec{r})$$

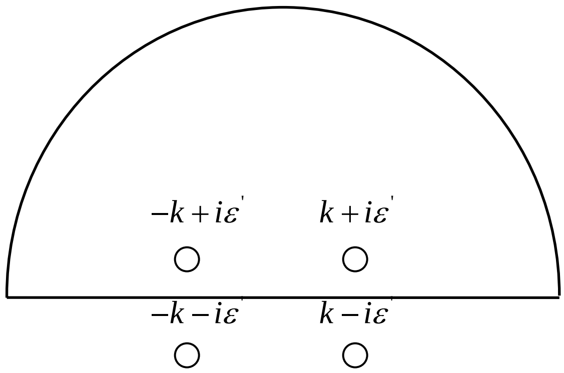
$$G_E^{(\pm)}\left(\vec{r},\vec{r}^{'}\right)=\lim_{\varepsilon\to 0^+}\int dE^{'}\,\frac{\phi_{E^{'}}\left(\vec{r}\right)\phi_{E^{'}}^{*}\left(\vec{r}^{'}\right)}{E^{'}-\left(E\pm i\varepsilon\right)}$$

$$=\lim_{\varepsilon\to 0^+}\int d^3k^{'}\cdot\frac{\dfrac{1}{\left(2\pi\right)^3}\exp\left[i\vec{k}^{'}\cdot\left(\vec{r}-\vec{r}^{'}\right)\right]}{\dfrac{\hbar^2k^{'2}}{2\mu}-\left(\dfrac{\hbar^2k^2}{2\mu}\pm i\varepsilon\right)}$$

$$=\lim_{\varepsilon\to 0^+}\frac{1}{\left(2\pi\right)^3}\cdot\frac{2\mu}{\hbar^2}\cdot\int\frac{d^3k^{'}e^{i\vec{k}^{'}\cdot\left(\vec{r}-\vec{r}^{'}\right)}}{k^{'2}-\left(k\pm i\varepsilon\right)^2}$$

$$=\lim_{\varepsilon\to 0^+}\frac{1}{\left(2\pi\right)^3}\cdot\frac{2\mu}{\hbar^2}\cdot 2\pi\cdot\int_0^\infty\frac{k^{'2}dk^{'}}{k^{'2}-\left(k\pm i\varepsilon\right)^2}\int_{-1}^{1}d\mu e^{ik^{'}\left|\vec{r}-\vec{r}^{'}\right|\mu}\qquad\left(\mu=\cos\theta\right)$$

$$=\frac{1}{\left(2\pi\right)^2}\cdot\frac{2\mu}{\hbar^2}\cdot\frac{1}{i\left|\vec{r}-\vec{r}^{'}\right|}\lim_{\varepsilon\to 0^+}\int_0^\infty k^{'}dk^{'}\,\frac{e^{ik^{'}\left|\vec{r}-\vec{r}^{'}\right|}-e^{ik^{'}\left|\vec{r}-\vec{r}^{'}\right|}}{k^{'2}-\left(k\pm i\varepsilon\right)^2}$$



$$=\frac{1}{\left(2\pi\right)^2}\cdot\frac{2\mu}{\hbar^2}\cdot\frac{1}{i\left|\vec{r}-\vec{r}^{'}\right|}\lim_{\varepsilon\to 0^+}\int_{-\infty}^{\infty}dk^{'}\,\frac{k^{'}e^{ik^{'}\left|\vec{r}-\vec{r}^{'}\right|}}{\left[k^{'}-\left(k\pm i\varepsilon\right)\right]\left[k^{'}+\left(k\pm i\varepsilon\right)\right]}$$

$$\oint_c=2\pi i\times\text{ Residue at }R^{'}=R+i\varepsilon^{'}$$

$$\left[\Rightarrow\frac{k+i\varepsilon^{'}}{2\left(k+i\varepsilon^{'}\right)}e^{i\left(k+i\varepsilon\right)\left|\vec{r}-\vec{r}^{'}\right|}\right]$$

$$=\pi i e^{i\left(k+i\varepsilon\right)\left|\vec{r}-\vec{r}^{'}\right|}$$

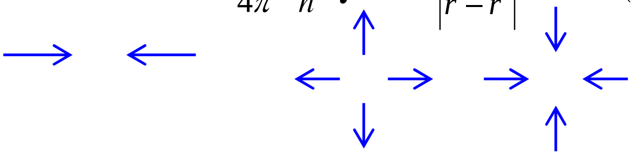
$$=\frac{1}{\left(2\pi\right)^2}\cdot\frac{2\mu}{\hbar^2}\cdot\frac{1}{i\left|\vec{r}-\vec{r}^{'}\right|}\pi i e^{\pm ik\left|\vec{r}-\vec{r}^{'}\right|}$$

$$=\frac{1}{4\pi}\cdot\frac{2\mu}{\hbar^2}\cdot\frac{e^{\pm ik\left|\vec{r}-\vec{r}^{'}\right|}}{\left|\vec{r}-\vec{r}^{'}\right|}$$

$$G_E\left(\vec{r},\vec{r}^{\,'}\right)=\alpha G_E^{(+)}+\beta G_E^{(-)}$$

$$=\frac{1}{4\pi}\cdot\frac{2\mu}{\hbar^2}\cdot\frac{\alpha e^{ik\left|\vec{r}-\vec{r}^{\,'}\right|}+\beta e^{-ik\left|\vec{r}-\vec{r}^{\,'}\right|}}{\left|\vec{r}-\vec{r}^{\,'}\right|}\qquad(\alpha+\beta=1)$$

$$=Ae^{i\vec{k}\cdot\vec{r}}+Be^{-i\vec{k}\cdot\vec{r}}-\frac{1}{4\pi}\cdot\frac{2\mu}{\hbar^2}\int\frac{\alpha e^{ik\left|\vec{r}-\vec{r}^{\,'}\right|}+\beta e^{-ik\left|\vec{r}-\vec{r}^{\,'}\right|}}{\left|\vec{r}-\vec{r}^{\,'}\right|}V\left(\vec{r}^{\,'}\right)\psi\left(\vec{r}^{\,'}\right)d^3r'$$



Boundary condition

$$B=0,\beta=0\qquad(\alpha=1)$$

$$\therefore\psi\left(\vec{r}\right)=Ae^{i\vec{k}\cdot\vec{r}}-\frac{1}{4\pi}\cdot\frac{2\mu}{\hbar^2}\int\frac{e^{ik\left|\vec{r}-\vec{r}^{\,'}\right|}}{\left|\vec{r}-\vec{r}^{\,'}\right|}V\left(\vec{r}^{\,'}\right)\psi\left(\vec{r}^{\,'}\right)d^3r'$$

$$\frac{1}{\left|\vec{r}-\vec{r}^{\,'}\right|}\to\frac{1}{r}$$

$$\left|\vec{r}-\vec{r}^{\,'}\right|=r\left(1-\frac{2\hat{r}\cdot\vec{r}^{\,'}}{r}+\cdots\right)^{\frac{1}{2}}$$

$$=r-\hat{r}\cdot\vec{r}^{\,'}$$

$$\therefore\frac{e^{ik\left|\vec{r}-\vec{r}^{\,'}\right|}}{\left|\vec{r}-\vec{r}^{\,'}\right|}=\frac{1}{r}e^{ikr}e^{-i\vec{k}\cdot\vec{r}^{\,'}}\qquad k\hat{r}=\vec{k}_r$$

$$r\to\infty\Rightarrow Ae^{i\vec{k}\cdot\vec{r}}-\frac{1}{4\pi}\frac{2\mu}{\hbar^2}\frac{e^{ikr}}{r}\int e^{i\vec{k}_r\cdot\vec{r}^{\,'}}V\left(\vec{r}^{\,'}\right)\psi\left(\vec{r}^{\,'}\right)d^3r'$$

$$f\left(\vec{k},\vec{k}_r\right)=-\frac{1}{4\pi}\frac{2\mu}{\hbar^2}\int e^{-i\vec{k}_r\cdot\vec{r}^{\,'}}V\left(\vec{r}^{\,'}\right)\cdot\frac{\psi\left(\vec{r}^{\,'}\right)}{A}d^3r'$$

So far f is exact

$$\psi\left(\vec{r}\right)=Ae^{i\vec{k}\cdot\vec{r}}$$

$$f_{\text{1st Born approx}}\left(\vec{k},\vec{k}_r\right)=-\frac{1}{4\pi}\frac{2\mu}{\hbar^2}\int d^{\,i\left(\vec{k}-\vec{k}_r\right)\cdot\vec{r}^{\,'}}V\left(\vec{r}^{\,'}\right)d^3r'$$

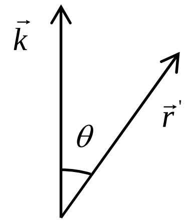
Summary

$$\psi(\vec r) = A\Bigg[\underbrace{e^{i\vec k\cdot\vec r}}_{\psi_{\text{inc}}} - \underbrace{\frac{1}{4\pi}\frac{2\mu}{\hbar^2}\int \frac{e^{ik|\vec r-\vec r'|}}{|\vec r-\vec r'|}V(\vec r')e^{i\vec k\cdot\vec r'}d^3r'}_{\psi_{\text{scatt}}}\Bigg]$$

1st Born Approx.

$$= A\left[e^{i\vec k\cdot\vec r} - \frac{e^{ikr}}{r}\frac{1}{4\pi}\cdot\frac{2\mu}{\hbar^2}\int e^{-i\vec k_r\cdot\vec r'}V(\vec r')e^{i\vec k\cdot\vec r'}d^3r'\right]$$

Validity for Born Approximation



$$|\psi_{\text{in}}(\vec r)| >> |\psi'_{\text{scatt}}| \qquad \text{neccessary condition}$$

$$\rightarrow |A|^2 >> |A|^2\left|-\frac{1}{4\pi}\frac{2\mu}{\hbar^2}\int\frac{e^{-i\vec k_r\cdot\vec r'}}{r'}V(\vec r')e^{i\vec k\cdot\vec r'}d^3r'\right|^2$$

$$\Rightarrow 1 >> \left|-\frac{1}{4\pi}\cdot\frac{2\mu}{\hbar^2}\cdot\frac{4\pi}{K}\int_0^\infty dr'\cdot\sin Kr'\cdot V(\vec r')\right| \qquad \vec K = \vec k - \vec k_r$$

Born Approximation is useful

when

(a) Velocity of incident particle is large

(b) weak int.