POSTECH 이성익 교수의

양자 세계에 관한 강연

- 9 ~ 10장 -

편집 도우미: POSTECH 학부생 임향택

Chapter 9 ~ 10

Matrix Representation of Operators & Spin

Heisenberg – Q.M. : array of numbers
Max Born – Matrices
Shrödinger equation 발견
Dirac – Abstract formulation

angular momentum

$$L^2|l,m\rangle = l(l+1)\hbar^2|l,m\rangle \qquad L^2 Y_{lm} = l(l+1)\hbar^2 Y_{lm}$$

$$L_z|l,m\rangle = m\hbar|l,m\rangle \qquad L_z Y_{lm} = m\hbar Y_{lm}$$

두 methods

$$L_z Y_{l,m} = m\hbar Y_{l,m}$$

$$\frac{\hbar}{i}\frac{\partial}{\partial\phi}Y_{l,m}(\theta,\phi) = m\hbar Y_{l,m}$$

Let $Y_{l,m}(\theta,\phi) = \Phi_{l,m}(\theta)\Phi_m(\phi)$

$$\frac{d\Phi_m(\phi)}{d\phi} = im\Phi_m(\phi)$$

$$\Phi_m(\phi) = \frac{1}{\sqrt{2\pi}}e^{im\phi}$$

$$\vdots$$

matrices로 표시

$$\langle l,m'|L_z|l,m\rangle = m\hbar\delta_{m',m}$$

$$\langle l,m'|L_\pm|l,m\rangle = \hbar\left[l(l+1)-m(m\pm1)\right]^{1/2}\delta_{m',m\pm1}$$

$l=1$인 경우

$$L_z = \hbar\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}, L_+ = \hbar\begin{pmatrix}0&\sqrt{2}&0\\0&0&\sqrt{2}\\0&0&0\end{pmatrix}, L_- = \hbar\begin{pmatrix}0&0&0\\\sqrt{2}&0&0\\0&\sqrt{2}&0\end{pmatrix}$$

Commutator relation Check

$$[L_+,L_-]=\hbar^2\begin{pmatrix}0&\sqrt{2}&0\\0&0&\sqrt{2}\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\\sqrt{2}&0&0\\0&\sqrt{2}&0\end{pmatrix}-\begin{pmatrix}0&0&0\\\sqrt{2}&0&0\\0&\sqrt{2}&0\end{pmatrix}\begin{pmatrix}0&\sqrt{2}&0\\0&0&\sqrt{2}\\0&0&0\end{pmatrix}$$

$$=\hbar^2\begin{pmatrix}2&0&0\\0&2&0\\0&0&0\end{pmatrix}-\hbar^2\begin{pmatrix}0&0&0\\0&2&0\\0&0&2\end{pmatrix}=2\hbar^2\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}=2\hbar L_z$$

그런데 문제점

$l=\frac{1}{2}$인 경우

$$\Theta_{\frac{1}{2},\frac{1}{2}}(\theta)=(\sin\theta)^{1/2}$$

$$\Phi_m(\phi)=e^{\pm\frac{i}{2}\phi}$$

$$\therefore Y_{\frac{1}{2},\pm\frac{1}{2}}=C_\pm\sqrt{\sin\theta}e^{\pm\frac{i\phi}{2}}$$

$$L_-Y_{\frac{1}{2},-\frac{1}{2}}=\hbar e^{-i\phi}\left(-\frac{\partial}{\partial\theta}+i\cot\theta\frac{\partial}{\partial\phi}\right)(\sin\theta)^l e^{il\phi}$$

$$=\hbar e^{-i\phi}\left(-\frac{\partial}{\partial\theta}+i\cot\theta\frac{\partial}{\partial\phi}\right)\sqrt{\sin\theta}e^{\frac{i\phi}{2}}$$

$$=\hbar e^{-i\phi}\left\{-\frac{1}{2}\frac{\cos\theta}{\sqrt{\sin\theta}}e^{\frac{i\phi}{2}}+i\cot\theta\sqrt{\sin\theta}\left(\frac{i}{2}\right)e^{\frac{i\phi}{2}}\right\}$$

$$=\hbar e^{-\frac{i\phi}{2}}\left(-\frac{1}{2}\frac{\cos\theta}{\sqrt{\sin\theta}}-\frac{1}{2}\frac{\cos\theta}{\sqrt{\sin\theta}}\right)$$

$$=-\hbar e^{-\frac{i\phi}{2}}\frac{\cos\theta}{\sqrt{\sin\theta}}$$

<u>Strange! 문제점</u>

$$\Theta_{\frac{1}{2},\frac{1}{2}}(\theta,\phi)=(\sin\theta)^{1/2}e^{\frac{i}{2}\phi}$$ not single valued function

$[L_+,L_-]=2\hbar L_z$ 등등을 만족하는 matrix는?

$$S_+=\hbar\begin{pmatrix}0&1\\0&0\end{pmatrix},S_-=\hbar\begin{pmatrix}0&0\\1&0\end{pmatrix},S_z=\hbar\begin{pmatrix}\frac{1}{2}&1\\0&\frac{1}{2}\end{pmatrix}$$

$$\vec{S} = \frac{1}{2}\hbar\vec{\sigma}$$

where

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$ called Pauli's Spin matrices

$$\left[\sigma_{x,}\sigma_y\right] = 2i\sigma_z$$

$$\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = 1$$

$$\sigma_x\sigma_y = -\sigma_y\sigma_x$$

$$\sigma_y\sigma_z = -\sigma_z\sigma_y$$

$$\sigma_z\sigma_x = -\sigma_x\sigma_z$$

Spinor – two component column vector

S_z 의 eigenspinor

$$S_z\begin{pmatrix} u \\ v \end{pmatrix} = \pm\frac{1}{2}\hbar\begin{pmatrix} u \\ v \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix} \qquad \therefore u = 1, v = 0$$

$$\therefore \chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

S_x 의 eigenspinor

$$S_x\begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{2}\hbar\begin{pmatrix} u \\ v \end{pmatrix}$$

$$\frac{1}{2}\hbar\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{2}\hbar\begin{pmatrix} u \\ v \end{pmatrix}$$

$$\therefore \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix} \qquad \therefore |u|^2 + |v|^2 = 1$$

$$u = v$$

$\therefore \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$는 S_x의 spin $\frac{1}{2}\hbar$ 상태

$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$는 S_x의 spin $-\frac{1}{2}\hbar$ 상태이다.

문제. $\hat{n}$ Vector에 대한 eigenvector

$$\hat{n} = n_x\hat{x} + n_y\hat{y} + n_z\hat{z}$$
$$= \sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y} + \cos\phi\hat{z}$$

$$\left(\vec{S}\cdot\hat{n}\right)\begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{2}\hbar\begin{pmatrix} u \\ v \end{pmatrix}$$

$$\left(S_x\cdot\sin\theta\cos\phi + S_y\cdot\sin\theta\sin\phi + S_z\cdot\cos\theta\right)\begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{2}\hbar\begin{pmatrix} u \\ v \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\sin\theta\cos\phi + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\sin\theta\sin\phi + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\cos\theta$$
$$= \begin{pmatrix} \cos\theta & \sin\theta\cos\phi - i\sin\theta\sin\phi \\ \sin\theta\cos\phi + i\sin\theta\sin\phi & -\cos\theta \end{pmatrix}$$
$$= \begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}$$

$$\begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}$$

$$\therefore \cos\theta\cdot u + \sin\theta\cdot e^{-i\phi}\cdot v = u$$
$$\sin\theta\cdot e^{i\phi}\cdot u + \cos\theta\cdot v = v$$
$$\left(\cos\theta - 1\right)u + \sin\theta\cdot e^{-i\phi}v = 0$$
$$\sin\theta\cdot e^{i\phi}\cdot u - \left(1+\cos\theta\right)\cdot v = 0$$

$$1 - \cos\theta = 2\sin^2\frac{\theta}{2}$$
$$\sin\theta = 2\sin\frac{\theta}{2}\cdot\cos\frac{\theta}{2}$$

$$\sin\frac{\theta}{2}\cdot u + \cos\frac{\theta}{2}\cdot e^{-i\phi}v = 0$$

$$v = -\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\cdot e^{i\phi}u = -\tan\frac{\theta}{2}\cdot e^{i\phi}u$$

$$
\begin{aligned}
1 &= |u|^2 + |v|^2 \\
&= |u|^2 + \tan^2\frac{\theta}{2}|u|^2 \\
&= \left(1+\tan^2\frac{\theta}{2}\right)|u|^2 \\
&= \sec^2\frac{\theta}{2}|u|^2
\end{aligned}
$$

$$\therefore\ u = \cos\frac{\theta}{2}$$

$$
\begin{aligned}
v &= -\tan\frac{\theta}{2}\cdot e^{i\phi}\cdot\cos\frac{\theta}{2} \\
&= -\sin\frac{\theta}{2}\cdot e^{i\phi}
\end{aligned}
$$

$$\therefore\ u = \begin{pmatrix} \cos\frac{\theta}{2} \\ -\sin\frac{\theta}{2}e^{i\phi} \end{pmatrix}$$

$\theta \to \theta + 2\pi$,

$$u = \begin{pmatrix} \cos\left(\frac{\theta}{2}+\pi\right) \\ -\sin\left(\frac{\theta}{2}+\pi\right)e^{i\phi} \end{pmatrix} = \begin{pmatrix} -\cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}e^{i\phi} \end{pmatrix}$$

문제. Spin $\vec{S}$가 $\vec{B}$ field 내에 있을 때 equation of motion을 구하여라.

$$\vec{M} = -\frac{eg}{2mc}\vec{S}, \qquad \vec{S} = \frac{\hbar}{2}\vec{\sigma}, \qquad g = 2\left(1+\frac{\alpha}{2\pi}+\cdots\right) = 2.0023192$$

$$
\begin{aligned}
H &= -\vec{M}\cdot\vec{B} \\
&= -\left(-\frac{eg}{2mc}\vec{S}\right)\cdot\vec{B} \\
&= \frac{eg}{2mc}\vec{S}\cdot\vec{B} \\
&= \frac{eg}{2mc}\cdot\frac{\hbar}{2}\vec{\sigma}\cdot\vec{B} \\
&= \frac{eg\hbar}{4mc}\vec{\sigma}\cdot\vec{B}
\end{aligned}
$$

$$H\psi = E\psi = i\hbar\frac{\partial}{\partial t}\psi$$

$$i\hbar\frac{d\psi(t)}{dt} = \frac{eg\hbar}{4mc}\vec{\sigma}\cdot\vec{B}\psi(t)$$

$$\psi(t) = \begin{bmatrix}\alpha_+(t)\\ \alpha_-(t)\end{bmatrix} = e^{-i\omega t}\begin{pmatrix}\alpha_+\\ \alpha_-\end{pmatrix}$$

$$i\hbar(-i\omega)e^{-i\omega t}\begin{pmatrix}\alpha_+\\ \alpha_-\end{pmatrix} = \frac{eg\hbar B}{4mc}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\alpha_+\\ \alpha_-\end{pmatrix}e^{-i\omega t}$$

$$\therefore \hbar\omega\begin{pmatrix}\alpha_+\\ \alpha_-\end{pmatrix} = \frac{eg\hbar B}{4mc}\begin{pmatrix}\alpha_+\\ -\alpha_-\end{pmatrix}$$

$$\therefore \begin{pmatrix}\alpha_+\\ \alpha_-\end{pmatrix} = \begin{pmatrix}1\\ 0\end{pmatrix} \quad \text{and} \quad \omega = \frac{egB}{4mc}$$

$$\therefore \psi(t) = \begin{bmatrix}\alpha_+(t)\\ \alpha_-(t)\end{bmatrix} = e^{-i\omega t}\begin{pmatrix}\alpha_+\\ \alpha_-\end{pmatrix} = e^{-i\omega t}\begin{pmatrix}1\\ 0\end{pmatrix} \quad \text{for} \quad \omega = \frac{egB}{4mc}$$

$$\psi(t) = \begin{bmatrix}\alpha_+(t)\\ \alpha_-(t)\end{bmatrix} = e^{-i\omega t}\begin{pmatrix}\alpha_+\\ \alpha_-\end{pmatrix} = e^{-i\omega t}\begin{pmatrix}0\\ 1\end{pmatrix} \quad \text{for} \quad \omega = -\frac{egB}{4mc}$$

Let $\psi(0) = \begin{pmatrix}a\\ b\end{pmatrix}$,

then $\psi(t) = \begin{pmatrix}ae^{-i\omega t}\\ be^{i\omega t}\end{pmatrix}$ for $\omega = \frac{egB}{4mc}$

If $t=0$, S는 S_x의 +방향 eigenstate이다.

$$\begin{pmatrix}a\\ b\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix}$$

$$\langle S_x \rangle = \frac{1}{2}\hbar \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} e^{i\omega t} & e^{-i\omega t} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} e^{-i\omega t} \\ e^{i\omega t} \end{pmatrix}$$

$$= \frac{1}{4}\hbar\begin{pmatrix} e^{i\omega t} & e^{-i\omega t} \end{pmatrix}\begin{pmatrix} e^{i\omega t} \\ e^{-i\omega t} \end{pmatrix}$$

$$= \frac{1}{4}\hbar\left(e^{2i\omega t} + e^{-2i\omega t}\right)$$

$$= \frac{1}{2}\hbar \cdot \cos 2\omega t$$

$$\langle S_y \rangle = \frac{1}{2}\hbar \cdot \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} e^{i\omega t} & e^{-i\omega t} \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} e^{-i\omega t} \\ e^{i\omega t} \end{pmatrix}$$

$$= \frac{\hbar}{2}\sin 2\omega t$$

Spin processes about the z-axis

$$2\omega = \frac{egB}{2mc} \simeq \frac{eB}{mc}$$

어떻게 잴 수 있나

Paramagnetic resonance

B_0 : pointing in the z-direction

$B_1 \cos\omega t$: small oscillation pointing in the x-direction

$$i\hbar\frac{d}{dt}\begin{bmatrix} a(t) \\ b(t) \end{bmatrix} = \frac{eg\hbar}{4mc}\begin{pmatrix} B_0 & B_1\cos\omega t \\ B_1\cos\omega t & -B_0 \end{pmatrix}\begin{pmatrix} a(t) \\ b(t) \end{pmatrix}$$

$$\omega_0 = \frac{egB_0}{4mc}, \quad \omega_1 = \frac{egB_1}{4mc}$$

$$i\frac{d}{dt}a(t) = \omega_0 a(t) + \omega_1 \cos\omega t \cdot b(t)$$

$$i\frac{d}{dt}b(t) = \omega_1 \cos\omega t \cdot a(t) - \omega_0 b(t)$$

Let
$$A(t) = a(t)e^{i\omega_0 t}$$
$$B(t) = b(t)e^{-i\omega_0 t}$$

$$\therefore \quad i\frac{dA(t)}{dt} = \omega_1 \cos\omega t \cdot B(t)e^{2i\omega_0 t}$$

$$\approx \frac{1}{2}\omega_1 e^{i(2\omega_0 - \omega)t}B(t)$$

$$i\frac{dB(t)}{dt} = \omega_1 \cos\omega t \cdot A(t) e^{-2i\omega_0 t}$$

$$\approx \frac{1}{2}\omega_1 e^{-i(2\omega_0-\omega)t} A(t)$$

$$B(t) = \frac{2i}{\omega_1}\cdot\frac{dA(t)}{dt}\cdot e^{-i(2\omega_0-\omega)t}$$

2nd order differential equation for $A(t)$

$$\frac{d^2A(t)}{dt^2} - i(2\omega_0-\omega)\frac{dA(t)}{dt} + \frac{\omega_1^2}{4}A(t) = 0$$

$$A(t) = A(0)e^{i\lambda t}$$

$$-\lambda^2 + (2\omega_0-\omega)\lambda + \frac{\omega_1^2}{4} = 0$$

$$\therefore \lambda_\pm = \frac{2\omega_0 - \omega \pm \sqrt{(2\omega_0-\omega)^2+\omega_1^2}}{2}$$

$$A(t) = A_+ e^{i\lambda_+ t} + A_- e^{i\lambda_- t}$$

$$B(t) = -\frac{2}{\omega_1}e^{-i(2\omega_0-\omega)t}\left(\lambda_+ A_+ e^{i\lambda_+ t} + \lambda_- A_- e^{i\lambda_- t}\right)$$

$$a(t) = e^{-i\omega_0 t}\left(A_+ e^{i\lambda_+ t} + A_- e^{i\lambda_- t}\right)$$

$$b(t) = -\frac{2}{\omega_1}e^{-i(\omega_0-\omega)t}\left(\lambda_+ A_+ e^{i\lambda_+ t} + \lambda_- A_- e^{i\lambda_- t}\right)$$

$$|b(t)|^2 = \frac{\omega_1^2}{(2\omega_0-\omega)^2+\omega_1^2}\cdot\frac{1-\cos\sqrt{(2\omega_0-\omega)^2+\omega_1^2 t}}{2}$$

if $\omega_1 \ll \omega, \omega_0$ 이면 $b(t)$가 매우 작다.

만약 $\omega = 2\omega_0$ 이면 Probability는

$$|b(t)|^2 \to \frac{1-\cos\omega_1 t}{2}$$

up state로 흡수가 잘 된다. Resonance. ω_0, g를 잴 수 있다.

Spin processes about the z-axis

1) Spin addition
2) Spin + Orbital
3) General case

1) $S+S$

$$\left[S_{1x}, S_{1y}\right] = i\hbar S_{1z}$$
$$\left[S_{2x}, S_{2y}\right] = i\hbar S_{2z}$$
$$\left[\vec{S}_1, \vec{S}_2\right] = 0$$

Def. $\vec{S} = \vec{S}_1 + \vec{S}_2$

$$\begin{aligned}\left[S_x, S_y\right] &= \left[S_{1x} + S_{2x}, S_{1y} + S_{2y}\right] \\ &= \left[S_{1x}, S_{1y}\right] + \left[S_{2x}, S_{2y}\right] \\ &= i\hbar S_{1z} + i\hbar S_{2z} \\ &= i\hbar S_z\end{aligned}$$

$\therefore \vec{S}$는 total spin angular momenta이다.

S^2과 S_z의 eigenfunction을 구해보자.

두 개의 spin system은 네 개의 state를 갖고 있다.

$$S_1^2 \chi_\pm^{(1)} = \frac{1}{2}\left(\frac{1}{2}+1\right)\hbar^2 \chi_\pm^{(1)}, S_2^2 \chi_\pm^{(2)} = \frac{1}{2}\left(\frac{1}{2}+1\right)\hbar^2 \chi_\pm^{(2)},$$

$$S_{1z}\chi_\pm^{(1)} = \pm\frac{1}{2}\hbar\chi_\pm^{(1)}, S_{2z}\chi_\pm^{(2)} = \pm\frac{1}{2}\hbar\chi_\pm^{(2)}$$

네 개의 state

$$\chi_+^{(1)}\chi_+^{(2)}, \chi_+^{(1)}\chi_-^{(2)}, \chi_-^{(1)}\chi_+^{(2)}, \chi_-^{(1)}\chi_-^{(2)}$$

이 모든 state는 S_z의 eigenstate이지만,

S^2의 eigenstate는 아니다.

$$S_z\chi_+^{(1)}\chi_+^{(2)}=\left(S_{1z}+S_{2z}\right)\chi_+^{(1)}\chi_+^{(2)}$$
$$=\left(\frac{1}{2}\hbar\chi_+^{(1)}\right)\chi_+^{(2)}+\chi_+^{(1)}\left(\frac{1}{2}\hbar\chi_+^{(2)}\right)$$
$$=\left(\frac{1}{2}\hbar+\frac{1}{2}\hbar\right)\chi_+^{(1)}\chi_+^{(2)}$$
$$=\hbar\chi_+^{(1)}\chi_+^{(2)}$$

$$S^2\chi_+^{(1)}\chi_+^{(2)}=\left(S_x^2+S_y^2+S_z^2\right)\chi_+^{(1)}\chi_+^{(2)}$$

$$S_+=S_x+iS_y \qquad S_x=\frac{1}{2}\left(S_++S_-\right)$$
$$\therefore$$
$$S_-=S_x-iS_y \qquad S_y=\frac{1}{2i}\left(S_+-S_-\right)$$

$$=\left[\frac{1}{4}\left(S_+^2+S_-^2+S_+S_-+S_-S_+-S_+^2-S_-^2+S_+S_-+S_-S_+\right)+S_z^2\right]\chi_+^{(1)}\chi_+^{(2)}$$
$$=\left[\frac{1}{2}\left(S_+S_-+S_-S_+\right)+S_z^2\right]\chi_+^{(1)}\chi_+^{(2)}$$

$$S^2=\left(S_1+S_2\right)^2$$
$$=S_1^2+S_2^2+S_1\cdot S_2+S_2\cdot S_1$$

$$S^2\chi_+^{(1)}\chi_+^{(2)}=\left(S_1^2+S_2^2+2S_{1z}S_{2z}+S_{1+}S_{2-}+S_{1-}S_{2+}\right)\chi_+^{(1)}\chi_+^{(2)}$$
$$=\left(\frac{3}{4}\hbar^2+\frac{3}{4}\hbar^2+2\cdot\frac{1}{2}\hbar\cdot\frac{1}{2}\hbar\right)\chi_+^{(1)}\chi_+^{(2)}$$
$$=2\hbar^2\chi_+^{(1)}\chi_+^{(2)}$$
$$=1\cdot\left(1+1\right)\hbar^2\chi_+^{(1)}\chi_+^{(2)}$$

$$S^2\chi_+^{(1)}\chi_-^{(2)}=\left(S_1^2+S_2^2+2S_{1z}S_{2z}+\underline{S_{1+}S_{2-}}+\underline{S_{1-}S_{2+}}\right)\chi_+^{(1)}\chi_-^{(2)}$$

이곳에서 문제된다

eigenvector가 아니다

$$\chi_+^{(1)}\chi_+^{(2)}=\left|1,1\right\rangle$$

Lowering

$$S_-\chi_+^{(1)}\chi_+^{(2)} = \left(S_{1-}\chi_+^{(1)}\right)\chi_+^{(2)} + \chi_+^{(1)}S_{2-}\chi_+^{(2)}$$
$$= \hbar\chi_-^{(1)}\chi_+^{(2)} + \hbar\chi_+^{(1)}\chi_-^{(2)}$$
$$= \hbar\left(\chi_-^{(1)}\chi_+^{(2)} + \chi_+^{(1)}\chi_-^{(2)}\right)$$
$$\therefore |1,0\rangle = \frac{1}{\sqrt{2}}\left(\chi_-^{(1)}\chi_+^{(2)} + \chi_+^{(1)}\chi_-^{(2)}\right)$$

$$L_+|l,m\rangle = \hbar\left[l(l+1) - m(m+1)\right]^{1/2}\cdot|l,m+1\rangle$$
$$L_-|l,m\rangle = \hbar\left[l(l+1) - m(m-1)\right]^{1/2}\cdot|l,m-1\rangle$$

$$S_-\chi_+^{(1)}\chi_+^{(2)} = S_-|1,1\rangle$$
$$= \hbar\left[1\cdot 2 - 1\cdot 0\right]^{1/2}|1,0\rangle$$
$$= \sqrt{2}\hbar|1,0\rangle$$
$$\therefore |1,0\rangle = \frac{1}{\sqrt{2}}\left(\chi_-^{(1)}\chi_+^{(2)} + \chi_+^{(1)}\chi_-^{(2)}\right)$$

Lower one more

$$|1,-1\rangle = \chi_-^{(1)}\chi_-^{(2)}$$

$$|1,1\rangle = \chi_+^{(1)}\chi_+^{(2)}$$
$$\therefore |1,0\rangle = \frac{1}{\sqrt{2}}\left(\chi_+^{(1)}\chi_-^{(2)} + \chi_-^{(1)}\chi_+^{(2)}\right)$$
$$|1,-1\rangle = \chi_-^{(1)}\chi_-^{(2)}$$

Let $|\psi\rangle = \alpha\chi_+^{(1)}\chi_-^{(2)} + \beta\chi_-^{(1)}\chi_+^{(2)}$

then $0 = \langle 1,0|\psi\rangle$

$$\therefore \alpha + \beta = 0$$
$$\therefore |\psi\rangle = \frac{1}{\sqrt{2}}\left(\chi_+^{(1)}\chi_-^{(2)} - \chi_-^{(1)}\chi_+^{(2)}\right)$$

Let $S_z|\psi\rangle = \left(S_{1z} + S_{2z}\right)\cdot\frac{1}{\sqrt{2}}\left(\chi_+^{(1)}\chi_-^{(2)} - \chi_-^{(1)}\chi_+^{(2)}\right)$

$$= 0$$

$$S^2|\psi\rangle = 0$$

$$\therefore |\psi\rangle = |0,0\rangle$$

$$\left[H,S_1^2\right]=\left[H,S_{1z}\right]=0$$
$$\left[H,S_2^2\right]=\left[H,S_{2z}\right]=0$$

$\Leftrightarrow$

$$\left[H,S^2\right]=\left[H,S_z\right]=0$$
$$\left[H,S_1^2\right]=\left[H,S_2^2\right]=0$$

Simultaneous eigen function of $H,S_1^2,S_2^2,S_{1z},S_{2z}$를 얻을 수 있다.

Simultaneous eigen function of H,S^2,S_z,S_1^2,S_2^2를 얻을 수 있다.

[문제]. 두개의 electron 사이에 Spin에 관계되는 Potential이 있다.

$$V(r)=V_1(r)+\frac{1}{\hbar^2}\vec{S}_1\cdot\vec{S}_2V_2(r)$$

$$\left[H,S_{1z}\right]\neq 0,\left[H,S_{2z}\right]\neq 0$$

$\therefore$ 쉬운 문제가 아니다.

$$\vec{S}_1\cdot\vec{S}_2=\frac{1}{2}\left(S^2-S_1^2-S_2^2\right)$$

S^2,S_1^2,S_2^2,H 은 서로 Commute한다.

$$V(r)=V_1(r)+\frac{1}{\hbar^2}\cdot\frac{1}{2}\left(S^2-S_1^2-S_2^2\right)V_2(r)$$

$$=V_1(r)+\frac{1}{\hbar^2}\cdot\frac{1}{2}\cdot\hbar^2\begin{pmatrix}1\cdot 2-\frac{1}{2}\left(\frac{1}{2}+1\right)-\frac{1}{2}\left(\frac{1}{2}+1\right)\\ 0-\frac{3}{2}\end{pmatrix}V_2(r)$$

$$=V_1(r)+\frac{1}{2}\begin{pmatrix}2-\frac{3}{2}\\ 0-\frac{3}{2}\end{pmatrix}V_2(r)$$

$$=V_1(r)+\frac{1}{4}V_2(r)\quad\text{or}\quad V_1(r)-\frac{3}{4}V_2(r)$$

$$\therefore V(r)=V_1(r)+\frac{1}{4}\begin{pmatrix}1\\ -3\end{pmatrix}V_2(r)\qquad\begin{pmatrix}\text{S=1}\\ \text{S=0}\end{pmatrix}$$

$$J^2\psi_{j,m\pm\frac{1}{2}}=J^2\alpha Y_{lm}\chi_+ + J^2\beta Y_{lm+1}\chi_-$$
$$=\left(l^2+s^2+2L_zS_z+L_+S_-+L_-S_+\right)\alpha Y_{lm}\chi_+$$
$$+\left(l^2+s^2+2L_zS_z+L_+S_-+L_-S_+\right)\beta Y_{lm+1}\chi_-$$

$$\frac{1}{\hbar^2}\cdot J^2\psi_{j,m+\frac{1}{2}}=\left[l(l+1)+\frac{3}{4}+2\cdot m\cdot\left(\frac{1}{2}\right)\right]\alpha Y_{lm}\chi_+ +\left[l(l+1)+\frac{3}{4}+2\cdot(m+1)\left(-\frac{1}{2}\right)\right]\beta Y_{l,m+1}\chi_-$$

$$+\sqrt{(l+m+1)(l-m)}\cdot\alpha Y_{l,m+1}\chi_- +\sqrt{(l+m+1)(l-m)}\beta Y_{lm}\chi_+$$

$$=\left[\left(l(l+1)+\frac{3}{4}+m\right)\alpha+\sqrt{(l+m+1)(l-m)}\beta\right]Y_{lm}\chi_+$$

$$+\left[\left(l(l+1)+\frac{3}{4}-(m+1)\right)\beta+\sqrt{(l+m+1)(l-m)}\alpha\right]Y_{l,m+1}\chi_-$$

$$\therefore\quad \begin{aligned} j(j+1)\alpha&=\left(l(l+1)+\frac{3}{4}+m\right)\alpha+\sqrt{(l+m+1)(l-m)}\beta\\ j(j+1)\beta&=\left(l(l+1)+\frac{3}{4}-(m+1)\right)\beta+\sqrt{(l+m+1)(l-m)}\alpha\end{aligned}$$

$$\therefore\quad \begin{aligned} \left(j(j+1)-l(l+1)-\frac{3}{4}-m\right)\alpha&=\sqrt{(l+m+1)(l-m)}\beta\\ \left(j(j+1)-l(l+1)-\frac{3}{4}+(m+1)\right)\beta&=\sqrt{(l+m+1)(l-m)}\alpha\end{aligned}$$

서로 곱하자.

$$(l-m)(l+m+1)=\left[j(j+1)-l(l+1)-\frac{3}{4}-m\right]\times\left[j(j+1)-l(l+1)-\frac{3}{4}+m+1\right]$$

$$j(j+1)-l(l+1)-\frac{3}{4}=l \text{ or } -l-1 \quad \text{이면 만족}$$

$$\begin{aligned}0&=j(j+1)-l^2-l-\frac{3}{4}-l\\ &=j^2+j-l^2-2l-\frac{3}{4}\\ &=j^2+j+\frac{1}{4}-(l^2+2l+1)\\ &=\left(j+\frac{1}{2}\right)^2-(l+1)^2\\ &=\left(j+\frac{1}{2}+l+1\right)\left(j+\frac{1}{2}-l-1\right)\\ &=\left(j+l+\frac{3}{2}\right)\left(j-l-\frac{1}{2}\right)\end{aligned}$$

$$\therefore j=l+\frac{1}{2}$$

or

$$0 = j(j+1) - l(l+1) - \frac{3}{4} + (l+1)$$

$$= j^2 + j - l^2 - \cancel{l} - \frac{3}{4} + \cancel{l} + 1$$

$$= j^2 + j - l^2 + \frac{1}{4}$$

$$= \left(j + \frac{1}{2}\right)^2 - l^2$$

$$= \left(j + l + \frac{1}{2}\right)\left(j - l + \frac{1}{2}\right)$$

$$\therefore j = l - \frac{1}{2}$$

$$\therefore j = l + \frac{1}{2} \text{ or } j = l - \frac{1}{2}$$

then $\alpha = \sqrt{\frac{l+m+1}{2l+1}}, \ \beta = \sqrt{\frac{l-m}{2l+1}}$

$$\therefore \psi_{l+\frac{1}{2},m+\frac{1}{2}} = \sqrt{\frac{l+m+1}{2l+1}} Y_{l,m}\chi_+ + \sqrt{\frac{l-m}{2l+1}} Y_{l,m+1}\chi_-$$

$$\psi_{l-\frac{1}{2},m+\frac{1}{2}} = \sqrt{\frac{l-m}{2l+1}} Y_{l,m}\chi_+ + \sqrt{\frac{l+m+1}{2l+1}} Y_{l,m+1}\chi_-$$

General case

두 개의 angular moment의 합

$Y^{(1)}_{l_1 m_1}, Y^{(2)}_{l_2 m_2}$ $\qquad -l_1 \le m_1 \le l_1$

$\qquad -l_2 \le m_2 \le l_2$

easy fact

1) $J_z = L_{1z} + L_{2z}$

2) total angular momentum 중 maximum에

$Y^{(1)}_{l_1 m_1}, Y^{(2)}_{l_2 m_2}$가 포함되어 있다.

$$J^2 Y^{(1)}_{l_1 l_1}, Y^{(2)}_{l_2 l_2} = \left(L_1^2 + L_2^2 + 2L_{1z}L_{2z} + L_{1+}L_{2-} + L_{1-}L_{2+}\right) Y^{(1)}_{l_1 l_1}, Y^{(2)}_{l_2 l_2}$$

$$= \hbar^2 \left(l_1(l_1+1) + l_2(l_2+1) + 2l_1 l_2\right) Y^{(1)}_{l_1 l_1}, Y^{(2)}_{l_2 l_2}$$

$$= \hbar^2 (l_1 + l_2)(l_1 + l_2 + 1) Y^{(1)}_{l_1 l_1}, Y^{(2)}_{l_2 l_2}$$

$$\therefore j_{\max} = l_1 + l_2,$$

$j_{\min} = |l_1 - l_2|$임을 증명할 수 있다.

몇 개의 state로 갈라지나?

$L_1 \times L_2$	j	개수
$(2l_1+1)\times(2l_2+1)$	l_1+l_2	$2(l_1+l_2)+1$
	l_1+l_2-1	$2(l_1+l_2-1)+1$
	$\vdots$	$\vdots$
	l_1-l_2	$2(l_1-l_2)+1$

$\therefore$ total 개수

$$\sum_{n=0}^{2l_2}\left[2(l_1-l_2+n)+1\right] = 2(l_1-l_2)(2l_2+1)+(2l_2+1)+2\cdot\frac{2l_2(2l_2+1)}{2}$$
$$=(2l_2+1)\cdot\left[2(l_1-l_2)+1+2l_2\right]$$
$$=(2l_2+1)(2l_1-2l_2+1+2l_2)$$
$$=(2l_1+1)(2l_2+1)$$

예. Spin $\frac{1}{2}$인 입자 두 개의 합

triplet $S=1, S_z=\begin{pmatrix}1\\0\\-1\end{pmatrix}$ $\qquad \begin{matrix}\chi_+^{(1)}\chi_+^{(2)}\\ \frac{1}{\sqrt{2}}\left(\chi_+^{(1)}\chi_-^{(2)}+\chi_-^{(1)}\chi_+^{(2)}\right)\\ \chi_-^{(1)}\chi_-^{(2)}\end{matrix}$ $\qquad$ Symmetric

singlet $S=0, S_z=0$ $\qquad \frac{1}{\sqrt{2}}\left(\chi_+^{(1)}\chi_-^{(2)}-\chi_-^{(1)}\chi_+^{(2)}\right)$ $\qquad$ Antisymmetric

total wave function
= anti-symmetric
= space × spin
S × A
A × S

Space part

$$=u(r)$$
$$=R_{nlm}(r)Y_{lm}(\theta,\phi)$$

$$r \to r \qquad Y_{lm}(\theta,\phi) \to Y_{lm}(\pi-\theta,\phi+\pi)$$
$$\theta \to \pi-\theta \qquad = (-1)^l Y_{lm}(\theta,\phi)$$
$$\phi \to \phi+\pi$$

응용: highly unstable elementary particle

Yukawa의 π meson

1. π^+, π^0, π^- +: charge +e

2. Spin 0

질문: Parity가 even인가 odd인가?

π^-가 에너지를 잃다가 드디어

α에 잡힌다: liquid deuterium에 의해 에너지 잃다가 Bohr radius에 가장 낮게 걸린 후 n,n으로 분리된다.

그래서 핵반응 후 n과 n으로 분리된다.

$\pi^- + d \to \underline{n} + \underline{n}$

↓ angular momentum 1

d의 angular momentum 1

$\pi^- \to$ Spin 0

Orbital angular momentum zero

($\because$ Bohr, lowest)

$\therefore n+n$은 angular momentum 1인 상태이다.

Spin	Angular momentum
0 (A)	1 (A) – not possible
1 (S)	$\begin{pmatrix} 0 \\ 1\,(A) \\ 2 \end{pmatrix} \therefore$ total A

$\therefore$ angular momentum 1만 가능하고 따라서 odd parity만을 갖게 된다.

Spectroscopic notation

$$^{2S+1}L_J$$

$$^1S_0, {}^1P_1, {}^1D_2, {}^1F_3, \cdots$$

S, P, D:핵에서, 원자에서

molecule 또는 핵자의 결합상태