POSTECH 이성익 교수의
양자 세계에 관한 강연
- 7장 -

편집 도우미: POSTECH 학부생 정윤영

Chapter 7

Angular Momentum

$L \equiv \vec{r} \times \vec{p}$

the definition of Angular Momentum

$$L_x = y \cdot p_z - z \cdot p_y$$
$$L_y = z \cdot p_x - x \cdot p_z$$
$$L_z = x \cdot p_y - y \cdot p_x = \frac{\hbar}{i}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \right) = \frac{\hbar}{i}\frac{\partial}{\partial \phi}$$
$$\vec{L}^2 = L_x{}^2 + L_y{}^2 + L_z{}^2$$

Angular Momentum의 "ladder operator"를 다음과 같이 정의한다.

$$L_\pm \equiv L_x \pm iL_y$$
$$= \frac{\hbar}{i}\left(\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} \right) \pm i\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} \right) \right)$$
$$= \frac{\hbar}{i}\left(\pm iz\left(\frac{\partial}{\partial x} \pm i\frac{\partial}{\partial y} \right) \mp (x \pm iy)\frac{\partial}{\partial z} \right)$$
$$= \frac{\hbar}{i}\left(\pm i \cdot r\cos\theta\left(\sin\theta \cdot e^{\pm i\phi}\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta \cdot e^{\pm i\phi}\frac{\partial}{\partial \theta} \pm i\frac{e^{\pm i\phi}}{r \cdot \sin\theta}\frac{\partial}{\partial \phi} \right) \right.$$
$$\left. \mp i \cdot r\sin\theta \cdot e^{\pm\phi}\left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta} \right) \right)$$
$$= \hbar e^{\pm i\phi}\left(\pm\frac{\partial}{\partial \theta} + i\cot\theta\frac{\partial}{\partial \phi} \right)$$

$$\left[H,\vec{L}^2\right]=0 \qquad \left[H,L_z\right]=0$$

$$\left[\vec{L}^2,L_z\right]=0 \qquad \left[\vec{L}^2,L_x\right]=0 \qquad \left[\vec{L}^2,L_y\right]=0$$

$$\left[L_x,L_y\right]=\left[y\cdot p_z-z\cdot p_y,z\cdot p_x-x\cdot p_z\right]$$

$$=\left[y\cdot p_z,z\cdot p_x\right]+\left[z\cdot p_y,x\cdot p_z\right]-\left[y\cdot p_z,x\cdot p_z\right]-\left[z\cdot p_y,z\cdot p_x\right]$$

$$=\left[y\cdot p_z,z\cdot p_x\right]+\left[z\cdot p_y,x\cdot p_z\right]=y\left[p_z,z\right]p_x+p_y\left[z,p_z\right]x$$

$$=y\left(-i\hbar\right)p_x+p_y\left(i\hbar\right)x=i\hbar\left(x\cdot p_y-y\cdot p_x\right)$$

$$=i\hbar L_z$$

$$\left[L_z,L_x\right]=i\hbar L_y$$

$$\left[L_y,L_z\right]=i\hbar L_x$$

$$L_+\cdot L_-=\left(L_x+iL_y\right)\left(L_x+iL_y\right)=L_x{}^2+L_y{}^2+i\left[L_y,L_x\right]$$

$$=L_x{}^2+L_y{}^2+i\left(-i\hbar L_z\right)=L_x{}^2+L_y{}^2+\hbar L_z$$

$$=\vec{L}^2-L_z{}^2+\hbar L_z$$

$$L_-\cdot L_+=\vec{L}^2-L_z{}^2-\hbar L_z$$

$$\therefore\left[L_+,L_-\right]=L_+\cdot L_--L_-\cdot L_+=2\hbar L_z$$

$$\vec{L}^2$$

$$=L_+\cdot L_--\hbar L_z+L_z{}^2$$

$$=L_-\cdot L_++\hbar L_z+L_z{}^2$$

$\left[L_+, L_z\right]$

$= \left[L_x + iL_y, L_z\right] = \left[L_x, L_z\right] + i\left[L_y, L_z\right] = -i\hbar L_y + i\left(i\hbar L_x\right) = -\hbar\left(L_x + iL_y\right)$

$= -\hbar L_+$

$\left[L_z, L_+\right] = \hbar L_+$

$\phi(r)\cdot Y_{l,m}\left(\theta,\phi\right)$: Laplace equation의 solution

$\vec{L}^2 \cdot Y_{l,m}\left(\theta,\phi\right) = l\left(l+1\right)\hbar^2 \cdot Y_{l,m}\left(\theta,\phi\right)$

$L_z \cdot Y_{l,m}\left(\theta,\phi\right) = m\hbar \cdot Y_{l,m}\left(\theta,\phi\right)$

계산상 편의를 위하여 eigenvalue를 위와 같이 설정한다.

$L_z \cdot Y_{l,m}\left(\theta,\phi\right) = \frac{\hbar}{i}\frac{\partial}{\partial\phi} Y_{l,m}\left(\theta,\phi\right) = m\hbar \cdot Y_{l,m}\left(\theta,\phi\right)$

$\therefore \frac{\partial}{\partial\phi} Y_{l,m}\left(\theta,\phi\right) = im \cdot Y_{l,m}\left(\theta,\phi\right)$

$\therefore Y_{l,m}\left(\theta,\phi\right) \sim e^{im\phi}$

$Y_{l,m}\left(\theta,\phi\right) = Y_{l,m}\left(\theta,\phi+2\pi\right)$

$\therefore e^{im2\pi} = 1$

m is integer.

$\vec{L}^2 \cdot L_\pm \cdot Y_{l,m}\left(\theta,\phi\right) = L_\pm \cdot \vec{L}^2 \cdot Y_{l,m}\left(\theta,\phi\right) = L_\pm \cdot l\left(l+1\right)\hbar^2 \cdot Y_{l,m}\left(\theta,\phi\right)$

$= l\left(l+1\right)\hbar^2 \cdot L_\pm \cdot Y_{l,m}\left(\theta,\phi\right)$

$L_\pm \cdot Y_{l,m}\left(\theta,\phi\right)$은 $\vec{L}^2$의 eigenfunction이다.

$$\left[L_z, L_+\right] = L_z \cdot L_+ - L_+ \cdot L_z = \hbar L_+$$
$$\therefore L_z \cdot L_+ = L_+ \cdot L_z + \hbar L_+$$

$$\begin{aligned} L_z \cdot L_+ \cdot Y_{l,m}(\theta,\phi) &= \left(L_+ \cdot L_z + \hbar L_+\right) \cdot Y_{l,m}(\theta,\phi) \\ &= L_+ \cdot L_z \cdot Y_{l,m}(\theta,\phi) + \hbar L_+ \cdot Y_{l,m}(\theta,\phi) \\ &= L_+ \cdot m\hbar \cdot Y_{l,m}(\theta,\phi) + \hbar L_+ \cdot Y_{l,m}(\theta,\phi) \\ &= (m+1)\hbar \cdot L_+ \cdot Y_{l,m}(\theta,\phi) \end{aligned}$$

$$L_z \cdot L_- \cdot Y_{l,m}(\theta,\phi) = (m-1)\hbar \cdot L_- \cdot Y_{l,m}(\theta,\phi)$$

$L_\pm \cdot Y_{l,m}(\theta,\phi)$은 L_z의 eigenfunction이다.

$$\begin{aligned} \left\langle Y_{l,m}(\theta,\phi) \middle| \vec{L}^2 \middle| Y_{l,m}(\theta,\phi) \right\rangle &= l(l+1)\hbar^2 \\ &= \left\langle Y_{l,m}(\theta,\phi) \middle| \vec{L}^+ \cdot \vec{L} \middle| Y_{l,m}(\theta,\phi) \right\rangle \\ &= \left\langle \vec{L} \cdot Y_{l,m}(\theta,\phi) \middle| \vec{L} \cdot Y_{l,m}(\theta,\phi) \right\rangle \geq 0 \end{aligned}$$

$$\therefore l(l+1) \geq 0$$

$$\begin{aligned} &\left\langle L_\pm Y_{l,m}(\theta,\phi) \middle| L_\pm Y_{l,m}(\theta,\phi) \right\rangle \\ &= \left\langle Y_{l,m}(\theta,\phi) \middle| L_\mp \cdot L_\pm \middle| Y_{l,m}(\theta,\phi) \right\rangle \\ &= \left\langle Y_{l,m}(\theta,\phi) \middle| \vec{L}^2 - L_z{}^2 \mp \hbar L_z \middle| Y_{l,m}(\theta,\phi) \right\rangle \\ &= \left\langle Y_{l,m}(\theta,\phi) \middle| \vec{L}^2 \middle| Y_{l,m}(\theta,\phi) \right\rangle - \left\langle Y_{l,m}(\theta,\phi) \middle| L_z{}^2 \middle| Y_{l,m}(\theta,\phi) \right\rangle \mp \hbar \left\langle Y_{l,m}(\theta,\phi) \middle| L_z \middle| Y_{l,m}(\theta,\phi) \right\rangle \\ &= l(l+1)\hbar^2 - m^2\hbar^2 \mp m\hbar^2 \\ &= \hbar^2 \left(l(l+1) - m^2 \mp m\right) \geq 0 \end{aligned}$$

$$\therefore l(l+1)-m^2-m\geq 0 \qquad l(l+1)-m^2+m\geq 0$$

$$\therefore l(l+1)\geq m^2+m \qquad l(l+1)\geq m^2-m$$

$$\therefore m^2+m-l(l+1)\leq 0 \qquad m^2-m-l(l+1)\leq 0$$

$$\therefore (m+l+1)(m-l)\leq 0 \qquad (m-l-1)(m+l)\leq 0$$

$$\therefore -l-1\leq m\leq l \qquad -l\leq m\leq l+1$$

$$\therefore -l\leq m\leq l$$

m의 개수: 2l+1개

$$L_\pm\cdot Y_{l,m}(\theta,\phi)=C_\pm(l,m)\cdot Y_{l,m\pm1}(\theta,\phi)$$

$$\left\langle Y_{l,m}(\theta,\phi)\middle|L_\mp\cdot L_\pm\middle|Y_{l,m}(\theta,\phi)\right\rangle$$

$$=\left\langle L_\pm\cdot Y_{l,m}(\theta,\phi)\middle|L_\pm\cdot Y_{l,m}(\theta,\phi)\right\rangle$$

$$=\left|C_\pm(l,m)\right|^2\left\langle Y_{l,m\pm1}(\theta,\phi)\middle|Y_{l,m\pm1}(\theta,\phi)\right\rangle=\left|C_\pm(l,m)\right|^2$$

$$=\left\langle Y_{l,m}(\theta,\phi)\middle|\vec{L}^2-L_z{}^2\mp\hbar L_z\middle|Y_{l,m}(\theta,\phi)\right\rangle$$

$$=\left\langle Y_{l,m}(\theta,\phi)\middle|\vec{L}^2\middle|Y_{l,m}(\theta,\phi)\right\rangle-\left\langle Y_{l,m}(\theta,\phi)\middle|L_z{}^2\middle|Y_{l,m}(\theta,\phi)\right\rangle\mp\hbar\left\langle Y_{l,m}(\theta,\phi)\middle|L_z\middle|Y_{l,m}(\theta,\phi)\right\rangle$$

$$=l(l+1)\hbar^2-m^2\hbar^2\mp m\hbar^2$$

$$\therefore C_\pm(l,m)=\sqrt{l(l+1)\hbar^2-m^2\hbar^2\mp m\hbar^2}=\hbar\sqrt{l(l+1)-m(m\pm1)}$$

$$Y_{l,l}(\theta,\phi)=\Theta_{l,l}(\theta)\cdot e^{il\phi}$$

m의 최대값은 l

$$\therefore L_{+}\cdot Y_{l,l}(\theta,\phi)=0$$

$$\hbar e^{i\phi}\left(\frac{\partial}{\partial\theta}+i\cot\theta\frac{\partial}{\partial\phi}\right)\cdot Y_{l,l}(\theta,\phi)$$

$$=\hbar e^{i\phi}\left(\frac{\partial}{\partial\theta}+i\cot\theta\frac{\partial}{\partial\phi}\right)\cdot\Theta_{l,l}(\theta)\cdot e^{il\phi}$$

$$=\hbar e^{i\phi}\left(e^{il\phi}\frac{d}{d\theta}\Theta_{l,l}(\theta)+i\cot\theta\cdot\Theta_{l,l}(\theta)\frac{d}{d\phi}e^{il\phi}\right)$$

$$=\hbar e^{i\phi}\left(e^{il\phi}\frac{d}{d\theta}\Theta_{l,l}(\theta)+i\cot\theta\cdot\Theta_{l,l}(\theta)\cdot(il)\cdot e^{il\phi}\right)$$

$$=\hbar e^{i\phi+il\phi}\left(\frac{d}{d\theta}\Theta_{l,l}(\theta)-l\cot\theta\cdot\Theta_{l,l}(\theta)\right)=0$$

$$\therefore\frac{d\Theta_{l,l}(\theta)}{d\theta}=l\cot\theta\cdot\Theta_{l,l}(\theta)$$

$$\therefore\frac{d\Theta_{l,l}(\theta)}{\Theta_{l,l}(\theta)}=l\frac{\cos\theta}{\sin\theta}\cdot d\theta$$

$$\therefore\int\frac{d\Theta_{l,l}(\theta)}{\Theta_{l,l}(\theta)}=l\int\frac{\cos\theta}{\sin\theta}\cdot d\theta$$

$$\therefore\ln\left|\Theta_{l,l}(\theta)\right|=l\ln\left|\sin\theta\right|=\ln\left|\sin\theta\right|^{l}$$

$$\therefore\Theta_{l,l}(\theta)=\sin^{l}\theta$$

$$\therefore Y_{l,l}(\theta,\phi)=\sin^{l}\theta\cdot e^{il\phi}$$

$$Y_{0,0}(\theta,\phi)=\frac{1}{\sqrt{4\pi}}\qquad Y_{1,0}(\theta,\phi)=\sqrt{\frac{3}{4\pi}}\cos\theta\qquad Y_{1,1}(\theta,\phi)=-\sqrt{\frac{3}{8\pi}}e^{i\phi}\sin\theta$$

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