POSTECH 이성익 교수의
양자 세계에 관한 강연
- 6장 -

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Chapter 6

Operator Method in Quantum Mechanics

Harmonic Oscillator

$$H=\frac{p^2}{2m}+\frac{1}{2}mw^2x^2=\frac{1}{2}mw^2x^2+\frac{p^2}{2m}=w\left[\frac{1}{2}mwx^2+\frac{p^2}{2mw}\right]$$

$$w\left[\sqrt{\frac{mw}{2}}x-i\frac{p}{\sqrt{2mw}}\right]\left[\sqrt{\frac{mw}{2}}x+i\frac{p}{\sqrt{2mw}}\right]$$

$$=w\left[\frac{mwx^2}{2}+\frac{p^2}{2mw}\right]+iw\frac{xp-px}{2}=w\left[\frac{mwx^2}{2}+\frac{p^2}{2mw}\right]+iw\frac{[x,p]}{2}$$

$$=w\left[\frac{mwx^2}{2}+\frac{p^2}{2mw}\right]+iw\frac{i\hbar}{2}=w\left[\frac{mwx^2}{2}+\frac{p^2}{2mw}\right]-\frac{\hbar w}{2}$$

$$=H-\frac{\hbar w}{2}$$

$$A^+\equiv\sqrt{\frac{mw}{2}}x-i\frac{p}{\sqrt{2mw}}$$

$$A\equiv\sqrt{\frac{mw}{2}}x+i\frac{p}{\sqrt{2mw}}$$

A^+: Raising operator

A: Lowering operator

$$wA^+A=H-\frac{1}{2}\hbar w \qquad wAA^+=H+\frac{1}{2}\hbar w$$

$$\therefore\ H=A\cdot A^+w-\frac{1}{2}\hbar w$$

$$=A^+\cdot Aw+\frac{1}{2}\hbar w$$

$$\left[A, A^{+}\right]$$
$$= \left[\sqrt{\frac{mw}{2}}x + i\frac{p}{\sqrt{2mw}}, \sqrt{\frac{mw}{2}}x - i\frac{p}{\sqrt{2mw}}\right]$$
$$= -i\frac{xp}{2} + i\frac{px}{2} - i\frac{xp}{2} + i\frac{px}{2} = i\left[p, x\right] = \hbar$$

$$\left[H, A\right]$$
$$= \left[\frac{1}{2}\hbar w + wA^{+}A, A\right] = \left[wA^{+}A, A\right]$$
$$= w\left[A^{+}A, A\right] = wA^{+}\left[A, A\right] + w\left[A^{+}, A\right]A$$
$$= w(-\hbar)A = -\hbar wA$$

$$\left[H, A^{+}\right]$$
$$= \left[\frac{1}{2}\hbar w + wA^{+}A, A^{+}\right] = \left[wA^{+}A, A^{+}\right]$$
$$= w\left[A^{+}A, A^{+}\right] = wA^{+}\left[A, A^{+}\right] + w\left[A^{+}, A^{+}\right]A$$
$$= wA^{+}\hbar = \hbar wA^{+}$$

$$H \cdot A - A \cdot H = -\hbar wA$$
$$H \cdot u_E = Eu_E$$
$$\therefore \left(H \cdot A - A \cdot H\right) \cdot u_E = H \cdot A \cdot u_E - A \cdot H \cdot u_E = H \cdot A \cdot u_E - A \cdot Eu_E = -\hbar wA \cdot u_E$$
$$\therefore H \cdot A \cdot u_E = A \cdot Eu_E - \hbar wA \cdot u_E = \left(E - \hbar w\right) A \cdot u_E$$

$$H \cdot A^{+} - A^{+} \cdot H = \hbar wA^{+}$$
$$H \cdot u_E = Eu_E$$
$$\therefore \left(H \cdot A^{+} - A^{+} \cdot H\right) \cdot u_E = H \cdot A^{+} \cdot u_E - A^{+} \cdot H \cdot u_E = H \cdot A^{+} \cdot u_E - A^{+} \cdot Eu_E = \hbar wA^{+} \cdot u_E$$
$$\therefore H \cdot A^{+} \cdot u_E = A^{+} \cdot Eu_E + \hbar wA^{+} \cdot u_E = \left(E + \hbar w\right) A^{+} \cdot u_E$$

$A \cdot u_E$ 는 Hamiltonian의 eigenfunction이다.

(이 때의 eigenvalue는 $E - \hbar w$)

$A^+ \cdot u_E$ 는 Hamiltonian의 eigenfunction이다.

(이 때의 eigenvalue는 $E + \hbar w$)

$$H \cdot A \cdot u_E = \left(E - \hbar w\right) A \cdot u_E$$

$$H \cdot A^+ \cdot u_E = \left(E + \hbar w\right) A^+ \cdot u_E$$

Harmonic oscillator의 ground state eigenfunction을 U_0라고 하자. Ground state보다 상태가 더 낮아질 수 없으므로,

$$A \cdot U_0 = 0$$

$$\therefore H \cdot U_0 = \left(A^+ \cdot Aw + \frac{1}{2}\hbar w \right) \cdot U_0 = \frac{1}{2}\hbar w U_0$$

$\rightarrow$Harmonic oscillator ground state에서의 eigenvalue: $\frac{1}{2}\hbar w$

1^{st} excited state: $A^+ \cdot U_0$

일반적으로,

$$U_n = \frac{1}{\sqrt{n!}} \left(\frac{A^+}{\sqrt{\hbar}} \right)^n \cdot U_0$$

Ground state eigenfunctioin을 직접 구해보자.

$$\left(\sqrt{\frac{mw}{2}} x + i \frac{p}{\sqrt{2mw}} \right) u_0(x) = 0, \qquad p = \frac{\hbar}{i} \frac{d}{dx}$$

$$\therefore 0 = \left(\sqrt{\frac{mw}{2}} x + i \frac{1}{\sqrt{2mw}} \frac{\hbar}{i} \frac{d}{dx} \right) U_0(x)$$

$$= \left(\sqrt{\frac{mw}{2}} x + \frac{\hbar}{\sqrt{2mw}} \frac{d}{dx} \right) U_0(x)$$

$$\therefore 0 = \left(xU_0(x) + \frac{\hbar}{mw} \frac{dU_0(x)}{dx} \right)$$

$$\therefore -xU_0(x) = \frac{\hbar}{mw} \frac{dU_0(x)}{dx}$$

$$\therefore -\frac{mw}{\hbar} xdx = \frac{dU_0(x)}{U_0(x)}$$

$$\therefore -\frac{mw}{\hbar} \frac{x^2}{2} = \ln U_0(x) + C'$$

$$\therefore U_0(x) = Ce^{-\frac{mw}{2\hbar} x^2}$$

$$\left(\begin{aligned} & 1 = C^2 \int_{-\infty}^{+\infty} e^{-\frac{mw}{\hbar} x^2} dx = C^2 \sqrt{\frac{\hbar \pi}{mw}} \\ & \therefore C = \left(\frac{mw}{\hbar \pi} \right)^{\frac{1}{4}} \end{aligned} \right)$$

Heisenberg's Picture

$$i\hbar \frac{d}{dt} \psi(t) = H \cdot \psi(t)$$

$$\therefore \psi(t) = e^{\frac{-iHt}{\hbar}} \psi(0)$$

$$\left(e^{\frac{-iHt}{\hbar}} = \sum_{n=0}^{\infty} \frac{\left(\frac{-iHt}{\hbar} \right)^n}{n!} : \text{ time evolution operator} \right)$$

$$\langle A\rangle_t = \langle\psi(t)|A|\psi(t)\rangle = \langle\psi(t)|A\cdot\psi(t)\rangle \qquad \rightarrow \text{Schrödinger's Picture}$$

$$= \left\langle e^{\frac{-iHt}{\hbar}}\psi(0)\middle| A\cdot e^{\frac{-iHt}{\hbar}}\psi(0)\right\rangle = \langle\psi(0)|e^{\frac{iHt}{\hbar}}A\cdot e^{\frac{-iHt}{\hbar}}|\psi(0)\rangle$$

$$= \langle\psi(0)|A(t)|\psi(0)\rangle \qquad \rightarrow \text{Heisenberg's Picture}$$

$$= \langle A(t)\rangle_0$$

$$\left(A(t) = e^{\frac{iHt}{\hbar}}A(0)\cdot e^{\frac{-iHt}{\hbar}} \right)$$

$$\frac{d}{dt}A(t) = \frac{iH}{\hbar}e^{\frac{iHt}{\hbar}}A(0)e^{-\frac{iHt}{\hbar}} + e^{\frac{iHt}{\hbar}}A(0)\left(-\frac{iH}{\hbar}\right)e^{-\frac{iHt}{\hbar}}$$

$$= \frac{i}{\hbar}e^{\frac{iHt}{\hbar}}\left[H, A(0)\right]e^{-\frac{iHt}{\hbar}} = \frac{i}{\hbar}\left[H, A(0)\right]$$

Harmonic oscillator의 경우,

$$H = wA^+A + \frac{1}{2}\hbar w$$

$$H(t) = wA^+(t)\cdot A(t) + \frac{1}{2}\hbar w$$

$$\left[A(t), A^+(t)\right] = \hbar$$

$$\frac{d}{dt}A(t) = \frac{i}{\hbar}\left[H, A(t)\right]$$

$$= \frac{i}{\hbar}\left[wA^+A, A\right] = \frac{iw}{\hbar}\left\{A^+\left[A, A\right] + \left[A^+, A\right]A\right\}$$

$$= \frac{iw}{\hbar}\left\{-\hbar A\right\} = -iwA(t)$$

$$\frac{d}{dt}A^{+}(t) = iwA^{+}(t)$$

→operator의 differential equatioin?

$$\therefore A(t) = e^{-iwt}A(0)$$

$$\Rightarrow \sqrt{\frac{mw}{2}}x(t) + i\frac{p(t)}{\sqrt{2mw}} = e^{-iwt}\left\{\sqrt{\frac{mw}{2}}x(0) + i\frac{p(0)}{\sqrt{2mw}}\right\}$$

$$\therefore A^{+}(t) = e^{iwt}A^{+}(0)$$

$$\Rightarrow \sqrt{\frac{mw}{2}}x(t) - i\frac{p(t)}{\sqrt{2mw}} = e^{iwt}\left\{\sqrt{\frac{mw}{2}}x(0) - i\frac{p(0)}{\sqrt{2mw}}\right\}$$

$$2\sqrt{\frac{mw}{2}}x(t) = e^{-iwt}\left\{\sqrt{\frac{mw}{2}}x(0) + i\frac{p(0)}{\sqrt{2mw}}\right\} + e^{iwt}\left\{\sqrt{\frac{mw}{2}}x(0) - i\frac{p(0)}{\sqrt{2mw}}\right\}$$

$$\therefore x(t) = e^{-iwt}\left\{\frac{1}{2}x(0) + i\frac{p(0)}{2mw}\right\} + e^{iwt}\left\{\frac{1}{2}x(0) - i\frac{p(0)}{2mw}\right\}$$

$$= x(0)\frac{e^{iwt} + e^{-iwt}}{2} - i\frac{p(0)}{2mw}\left(e^{iwt} - e^{-iwt}\right)$$

$$= x(0)\frac{e^{iwt} + e^{-iwt}}{2} + \frac{p(0)}{mw}\frac{\left(e^{iwt} - e^{-iwt}\right)}{2i}$$

$$= x(0)\cos wt + \frac{p(0)}{mw}\sin wt$$

$$p(t) = p(0)\cos wt - mwx(0)\sin wt$$

→x(t) & p(t) are operators.
→시간에 따른 operator의 변화를 기술.