Quantum Mechanics in 15 Minutes

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Two Principles

Superposition Principle

Consider a quantum register of qubits, which we refer to them by symbol S.

In[ ]:=
```
Let[Qubit, S]
Let[Complex, c]
```

These are the states of the single qubit that can be physically discriminated.

In[ ]:=
```
bs = Basis[S]
```

Out[ ]=

$$\{|0_S\rangle, |1_S\rangle\}$$

Once we regard them physically relevant, then we also have to accept the following *superposition state* as physically valid.

In[ ]:=
```
new = bs.c@{0, 1}
```

Out[ ]=

$$c_0 |0_S\rangle + c_1 |1_S\rangle$$

In[ ]:=
```
bs = Basis[S@{1, 2}]
```

Out[ ]=

$$\{|0_{S_1} 0_{S_2}\rangle, |0_{S_1} 1_{S_2}\rangle, |1_{S_1} 0_{S_2}\rangle, |1_{S_1} 1_{S_2}\rangle\}$$

In[ ]:=
```
new = bs.c[{0, 1}, {0, 1}]
```

Out[ ]=

$$c_{0,0} |0_{S_1} 0_{S_2}\rangle + c_{0,1} |0_{S_1} 1_{S_2}\rangle + c_{1,0} |1_{S_1} 0_{S_2}\rangle + c_{1,1} |1_{S_1} 1_{S_2}\rangle$$

Complementarity Principle

Suppose that a quantum register of two qubits are in the following *superposition state*.

In[ ]:=
```
in = Basis[S@{1, 2}].c[{0, 1}, {0, 1}]
```

Out[ ]=

$$c_{0,0} |0_{S_1} 0_{S_2}\rangle + c_{0,1} |0_{S_1} 1_{S_2}\rangle + c_{1,0} |1_{S_1} 0_{S_2}\rangle + c_{1,1} |1_{S_1} 1_{S_2}\rangle$$

After a measurement of physical quantities, survive only those states that are consistent with the measurement outcomes.

In[∘]:=
```
Quiet[
 Echo[ops = S[{1, 2}, 3]];
 out = Measurement[ops] ** in;
 val = Readout[ops];
 val → out,
 Measurement::nonum
]
```

» $\{S_1^Z, S_2^Z\}$

Out[∘]=

$\{0, 1\} \to c_{0,1} \left|0_{S_1} 1_{S_2}\right\rangle$

Here is another example, where we measure the so-called *parity* $Z_1 \otimes Z_2$.

In[∘]:=
```
Quiet[
 Echo[ops = S[1, 3] ** S[2, 3]];
 out = Measurement[S[1, 3] ** S[2, 3]] ** in;
 val = Readout[S[1, 3] ** S[2, 3]];
 val → out,
 Measurement::nonum
]
```

» $S_1^Z S_2^Z$

Out[∘]=

$0 \to c_{0,0} \left|0_{S_1} 0_{S_2}\right\rangle + c_{1,1} \left|1_{S_1} 1_{S_2}\right\rangle$

Problems

1. Given a quantum state just before a specific measurement, can you calculate all possible measurement outcomes and the probability for each outcome?

Technical Rules

Quantum Systems

In this internship program, we will consider a collection of qubits, i.e., a *quantum register*. In Q3, a quantum register is referred to by a symbol; in this particular case, we have chosen `S`.

In[∘]:=
```
Let[Qubit, S]
```

For example, we will consider a quantum register of three qubits. The qubits are referred to by `S[k,$]`. for $k = 1, 2, 3$.

In[∘]:=
```
$n = 3;
kk = Range[$n];
sys = S[kk, $]
```

Out[∘]=

$\{S_1, S_2, S_3\}$

Computational Basis

Any quantum system is associated with a Hilbert space $\mathcal{H}$; that is, a vector space of quantum state vectors. The Hilbert space $\mathcal{H}$ may be specified by the *computational basis.*

In[◦]:=
```
bs = Basis[sys]
```
Out[◦]=

$$\{|0_{S_1}0_{S_2}0_{S_3}\rangle, |0_{S_1}0_{S_2}1_{S_3}\rangle, |0_{S_1}1_{S_2}0_{S_3}\rangle, |0_{S_1}1_{S_2}1_{S_3}\rangle, |1_{S_1}0_{S_2}0_{S_3}\rangle, |1_{S_1}0_{S_2}1_{S_3}\rangle, |1_{S_1}1_{S_2}0_{S_3}\rangle, |1_{S_1}1_{S_2}1_{S_3}\rangle\}$$

Recall that the computational basis is an *orthonormal basis*. This means that all elements in the basis may be physically distinguished from each other.

In[◦]:=
```
pp = Outer[Multiply, Dagger[bs], bs];
MatrixForm[pp, TableHeadings → {Dagger@bs, bs}]
```
Out[◦]//MatrixForm=

	$\vert 0_{S_1}0_{S_2}0_{S_3}\rangle$	$\vert 0_{S_1}0_{S_2}1_{S_3}\rangle$	$\vert 0_{S_1}1_{S_2}0_{S_3}\rangle$	$\vert 0_{S_1}1_{S_2}1_{S_3}\rangle$	$\vert 1_{S_1}0_{S_2}0_{S_3}\rangle$	$\vert 1_{S_1}0_{S_2}1_{S_3}\rangle$	$\vert 1_{S_1}1_{S_2}$
$\langle 0_{S_1}0_{S_2}0_{S_3}\vert$	1	0	0	0	0	0	0
$\langle 0_{S_1}0_{S_2}1_{S_3}\vert$	0	1	0	0	0	0	0
$\langle 0_{S_1}1_{S_2}0_{S_3}\vert$	0	0	1	0	0	0	0
$\langle 0_{S_1}1_{S_2}1_{S_3}\vert$	0	0	0	1	0	0	0
$\langle 1_{S_1}0_{S_2}0_{S_3}\vert$	0	0	0	0	1	0	0
$\langle 1_{S_1}0_{S_2}1_{S_3}\vert$	0	0	0	0	0	1	0
$\langle 1_{S_1}1_{S_2}0_{S_3}\vert$	0	0	0	0	0	0	1
$\langle 1_{S_1}1_{S_2}1_{S_3}\vert$	0	0	0	0	0	0	0

Quantum States

Quantum states are described by a vector belonging to the Hilbert space $\mathcal{H}$.

In[◦]:=
```
in = Ket[sys → {0, 0, 1}] - I * Ket[sys → {1, 1, 1}]
```
Out[◦]=

$$|0_{S_1}0_{S_2}1_{S_3}\rangle - i\,|1_{S_1}1_{S_2}1_{S_3}\rangle$$

Here, we have ignored normalization.

In[◦]:=
```
Dagger[in] ** in
```
Out[◦]=

2

Quantum Operations

Ideally, on a quantum computer, the time-evolution of quantum states (i.e., *computations*) are described by a *quantum operation*.

In[◦]:=
```
Let[Real, ϕ]
op = CNOT[S@{1, 2}, S[3]] ** Rotation[ϕ[1], S[1, 2]] **
    Rotation[ϕ[2], S[2, 1]] ** Rotation[ϕ[3], S[3, 3]] // Elaborate;
```

Ideally, every quantum operation on a quantum computer must be *unitary*.

In[∘]:=
```
EchoTiming[Dagger[op] ** op // Simplify]
```
3.01869

Out[∘]=

1

After the quantum operation, we have the result of the quantum computation at hand.

In[∘]:=
```
out = op ** in
```
Out[∘]=

$$\mathrm{Cos}\left[\frac{1}{2}(\phi_1-\phi_2)\right]\left|0_{S_1}0_{S_2}1_{S_3}\right\rangle\left(\mathrm{Cos}\left[\frac{\phi_3}{2}\right]+i\,\mathrm{Sin}\left[\frac{\phi_3}{2}\right]\right)+ i\left|0_{S_1}1_{S_2}1_{S_3}\right\rangle\mathrm{Sin}\left[\frac{1}{2}(\phi_1-\phi_2)\right]\left(\mathrm{Cos}\left[\frac{\phi_3}{2}\right]+i\,\mathrm{Sin}\left[\frac{\phi_3}{2}\right]\right)+ \left|1_{S_1}0_{S_2}1_{S_3}\right\rangle\mathrm{Sin}\left[\frac{1}{2}(\phi_1-\phi_2)\right]\left(\mathrm{Cos}\left[\frac{\phi_3}{2}\right]+i\,\mathrm{Sin}\left[\frac{\phi_3}{2}\right]\right)+ \mathrm{Cos}\left[\frac{1}{2}(\phi_1-\phi_2)\right]\left|1_{S_1}1_{S_2}0_{S_3}\right\rangle\left(-i\,\mathrm{Cos}\left[\frac{\phi_3}{2}\right]+\mathrm{Sin}\left[\frac{\phi_3}{2}\right]\right)$$

Measurement

After "computation", we need to check the result by performing measurements. Quantum computers can perform measurement only in the computational basis.

In[∘]:=
```
mm = Measurement[S[kk, 3]]
```
Out[∘]=

$$\mathrm{Measurement}\left[\left\{S_1^Z, S_2^Z, S_3^Z\right\}\right]$$

If this is really the end of your calculation, just checking the measurement outcomes is enough. The post-measurement state is of no concern.

In[∘]:=
```
Quiet[
 new = mm ** out;
 val = Readout[mm];
 val → new,
 Measurement::nonum
]
```
Out[∘]=

$$\{1, 1, 0\} \to -\frac{1}{2}\, i\, e^{-\frac{1}{2} i (\phi_1+\phi_2-\phi_3)}\left(e^{i\phi_1}+e^{i\phi_2}\right)\left|1_{S_1}1_{S_2}0_{S_3}\right\rangle$$

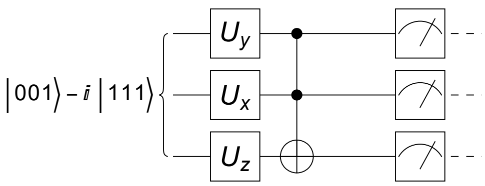
Problem: What if you want to make a measurement of operators other than the Pauli Z operator? For example, what if you want to measure $X\otimes I$, $I\otimes Y$, $Z\otimes Z$, or $X\otimes Z$?

Quantum Circuits

All the procedures described above may be described graphically using quantum circuits.

In[ ]:=
```
qc = QuantumCircuit[
  in,
  {Rotation[ϕ[1], S[1, 2]],
   Rotation[ϕ[2], S[2, 1]],
   Rotation[ϕ[3], S[3, 3]]},
  CNOT[S@{1, 2}, S[3]],
  "Spacer",
  mm,
  "PortSize" → {2.1, 1}]
```

Out[ ]=



In[ ]:=
```
Quiet[
 new = Elaborate[qc];
 val = Readout[Measurements@qc];
 val → new,
 Measurement::nonum]
```

Out[ ]=

$$\{1, 1, 0\} \rightarrow -\frac{1}{2}\, i\, e^{-\frac{1}{2} i (\phi_1+\phi_2-\phi_3)} \left(e^{i \phi_1} + e^{i \phi_2}\right) \left|1_{S_1} 1_{S_2} 0_{S_3}\right\rangle$$

More Technical Stuff

In[ ]:=
```
Let[Qubit, S]
```

Matrix representation of quantum states

In[ ]:=
```
bs = Basis[S]
```

Out[ ]=

$$\left\{\left|0_S\right\rangle, \left|1_S\right\rangle\right\}$$

In[ ]:=
```
mm = MatrixForm /@ Matrix[bs]
```

Out[ ]=

$$\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$$

In[ ]:=
```
Thread[bs → mm] // TableForm
```

Out[ ]//TableForm=

$$\left|0_S\right\rangle \rightarrow \begin{pmatrix}1\\0\end{pmatrix}$$

$$\left|1_S\right\rangle \rightarrow \begin{pmatrix}0\\1\end{pmatrix}$$

In[ ]:=
```
Let[Complex, c]
ket = bs.c[{0, 1}]
```
Out[ ]=

$$c_0 \left|0_S\right\rangle + c_1 \left|1_S\right\rangle$$

In[ ]:=
```
vec = Matrix[ket]
vec // MatrixForm
```
Out[ ]=

SparseArray[Specified elements: 2 Dimensions: {2}]

Out[ ]//MatrixForm=

$$\begin{pmatrix} c_0 \\ c_1 \end{pmatrix}$$

In[ ]:=
```
bs = Basis[S@{1, 2}]
```
Out[ ]=

$$\left\{ \left|0_{S_1} 0_{S_2}\right\rangle, \left|0_{S_1} 1_{S_2}\right\rangle, \left|1_{S_1} 0_{S_2}\right\rangle, \left|1_{S_1} 1_{S_2}\right\rangle \right\}$$

In[ ]:=
```
Thread[bs → MatrixForm /@ Matrix[bs]]
```
Out[ ]=

$$\left\{ \left|0_{S_1} 0_{S_2}\right\rangle \to \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \left|0_{S_1} 1_{S_2}\right\rangle \to \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \left|1_{S_1} 0_{S_2}\right\rangle \to \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \left|1_{S_1} 1_{S_2}\right\rangle \to \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

In[ ]:=
```
ket = bs[[1]] × c[1] + bs[[2]] * c[2] + bs[[4]] ** c[3]
```
Out[ ]=

$$c_1 \left|0_{S_1} 0_{S_2}\right\rangle + c_2 \left|0_{S_1} 1_{S_2}\right\rangle + c_3 \left|1_{S_1} 1_{S_2}\right\rangle$$

In[ ]:=
```
vec = Matrix[ket];
vec // MatrixForm
```
Out[ ]//MatrixForm=

$$\begin{pmatrix} c_1 \\ c_2 \\ 0 \\ c_3 \end{pmatrix}$$

Matrix representation of quantum operations

In[ ]:=
```
op = EulerRotation[{Pi / 3, Pi / 6, 0}, S[1]] **
   EulerRotation[{Pi / 2, Pi / 2, Pi / 3}, S[2]] ** CNOT[S[1], S[2]];
```

In[ ]:=
```
mat = Matrix[op, S@{1, 2}];
mat // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} -\frac{1}{4} i \sqrt{\frac{3}{2}} - \frac{\frac{1}{4}+\frac{i}{2}}{\sqrt{2}} & \left(-\frac{1}{4}+\frac{i}{4}\right)\sqrt{\frac{3}{2}} - \frac{\frac{1}{4}-\frac{i}{4}}{\sqrt{2}} & \left(\frac{1}{4}-\frac{i}{4}\right)\sqrt{\frac{3}{2}} - \frac{\frac{1}{4}-\frac{i}{4}}{\sqrt{2}} & -\frac{\sqrt{\frac{3}{2}}}{4} + \frac{\frac{1}{2}+\frac{i}{4}}{\sqrt{2}} \\ \frac{\sqrt{\frac{3}{2}}}{4} + \frac{\frac{1}{2}-\frac{i}{4}}{\sqrt{2}} & \left(\frac{1}{4}+\frac{i}{4}\right)\sqrt{\frac{3}{2}} + \frac{\frac{1}{4}+\frac{i}{4}}{\sqrt{2}} & \left(-\frac{1}{4}-\frac{i}{4}\right)\sqrt{\frac{3}{2}} + \frac{\frac{1}{4}+\frac{i}{4}}{\sqrt{2}} & -\frac{1}{4} i \sqrt{\frac{3}{2}} - \frac{\frac{1}{4}-\frac{i}{2}}{\sqrt{2}} \\ \left(\frac{1}{4}-\frac{i}{4}\right)\sqrt{\frac{3}{2}} - \frac{\frac{1}{4}-\frac{i}{4}}{\sqrt{2}} & \frac{1}{4} i \sqrt{\frac{3}{2}} - \frac{\frac{1}{4}+\frac{i}{2}}{\sqrt{2}} & -\frac{\sqrt{\frac{3}{2}}}{4} - \frac{\frac{1}{2}+\frac{i}{4}}{\sqrt{2}} & \left(\frac{1}{4}-\frac{i}{4}\right)\sqrt{\frac{3}{2}} + \frac{\frac{1}{4}-\frac{i}{4}}{\sqrt{2}} \\ \left(\frac{1}{4}+\frac{i}{4}\right)\sqrt{\frac{3}{2}} - \frac{\frac{1}{4}+\frac{i}{4}}{\sqrt{2}} & \frac{\sqrt{\frac{3}{2}}}{4} - \frac{\frac{1}{2}-\frac{i}{4}}{\sqrt{2}} & \frac{1}{4} i \sqrt{\frac{3}{2}} - \frac{\frac{1}{4}-\frac{i}{2}}{\sqrt{2}} & \left(\frac{1}{4}+\frac{i}{4}\right)\sqrt{\frac{3}{2}} + \frac{\frac{1}{4}+\frac{i}{4}}{\sqrt{2}} \end{pmatrix}$$

In[ ]:=
```
in = Ket[S@{1, 2}]
```

Out[ ]=

$$\left|0_{S_1} 0_{S_2}\right\rangle$$

In[ ]:=
```
out = op ** in
```

Out[ ]=

$$-\frac{i\left((2-i)+\sqrt{3}\right)\left|0_{S_1} 0_{S_2}\right\rangle}{4\sqrt{2}} + \frac{\left((2-i)+\sqrt{3}\right)\left|0_{S_1} 1_{S_2}\right\rangle}{4\sqrt{2}} + \frac{\left(\frac{1}{4}-\frac{i}{4}\right)\left(-1+\sqrt{3}\right)\left|1_{S_1} 0_{S_2}\right\rangle}{\sqrt{2}} + \frac{\left(\frac{1}{4}+\frac{i}{4}\right)\left(-1+\sqrt{3}\right)\left|1_{S_1} 1_{S_2}\right\rangle}{\sqrt{2}}$$

In[ ]:=
```
vin = Matrix[in, S@{1, 2}];
vin // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

In[ ]:=
```
mat // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} -\frac{1}{4} i \sqrt{\frac{3}{2}} - \frac{\frac{1}{4}+\frac{i}{2}}{\sqrt{2}} & \left(-\frac{1}{4}+\frac{i}{4}\right)\sqrt{\frac{3}{2}} - \frac{\frac{1}{4}-\frac{i}{4}}{\sqrt{2}} & \left(\frac{1}{4}-\frac{i}{4}\right)\sqrt{\frac{3}{2}} - \frac{\frac{1}{4}-\frac{i}{4}}{\sqrt{2}} & -\frac{\sqrt{\frac{3}{2}}}{4} + \frac{\frac{1}{2}+\frac{i}{4}}{\sqrt{2}} \\ \frac{\sqrt{\frac{3}{2}}}{4} + \frac{\frac{1}{2}-\frac{i}{4}}{\sqrt{2}} & \left(\frac{1}{4}+\frac{i}{4}\right)\sqrt{\frac{3}{2}} + \frac{\frac{1}{4}+\frac{i}{4}}{\sqrt{2}} & \left(-\frac{1}{4}-\frac{i}{4}\right)\sqrt{\frac{3}{2}} + \frac{\frac{1}{4}+\frac{i}{4}}{\sqrt{2}} & -\frac{1}{4} i \sqrt{\frac{3}{2}} - \frac{\frac{1}{4}-\frac{i}{2}}{\sqrt{2}} \\ \left(\frac{1}{4}-\frac{i}{4}\right)\sqrt{\frac{3}{2}} - \frac{\frac{1}{4}-\frac{i}{4}}{\sqrt{2}} & \frac{1}{4} i \sqrt{\frac{3}{2}} - \frac{\frac{1}{4}+\frac{i}{2}}{\sqrt{2}} & -\frac{\sqrt{\frac{3}{2}}}{4} - \frac{\frac{1}{2}+\frac{i}{4}}{\sqrt{2}} & \left(\frac{1}{4}-\frac{i}{4}\right)\sqrt{\frac{3}{2}} + \frac{\frac{1}{4}-\frac{i}{4}}{\sqrt{2}} \\ \left(\frac{1}{4}+\frac{i}{4}\right)\sqrt{\frac{3}{2}} - \frac{\frac{1}{4}+\frac{i}{4}}{\sqrt{2}} & \frac{\sqrt{\frac{3}{2}}}{4} - \frac{\frac{1}{2}-\frac{i}{4}}{\sqrt{2}} & \frac{1}{4} i \sqrt{\frac{3}{2}} - \frac{\frac{1}{4}-\frac{i}{2}}{\sqrt{2}} & \left(\frac{1}{4}+\frac{i}{4}\right)\sqrt{\frac{3}{2}} + \frac{\frac{1}{4}+\frac{i}{4}}{\sqrt{2}} \end{pmatrix}$$

In[∘]:=
```
out = op ** in;
vout = Matrix[out, S@{1, 2}];
vout // MatrixForm
```
Out[∘]//MatrixForm=

$$\begin{pmatrix} -\frac{1}{4}\,i\,\sqrt{\frac{3}{2}} - \frac{\frac{1}{4}+\frac{i}{2}}{\sqrt{2}} \\ \frac{\sqrt{\frac{3}{2}}}{4} + \frac{\frac{1}{2}-\frac{i}{4}}{\sqrt{2}} \\ \left(\frac{1}{4}-\frac{i}{4}\right)\sqrt{\frac{3}{2}} - \frac{\frac{1}{4}-\frac{i}{4}}{\sqrt{2}} \\ \left(\frac{1}{4}+\frac{i}{4}\right)\sqrt{\frac{3}{2}} - \frac{\frac{1}{4}+\frac{i}{4}}{\sqrt{2}} \end{pmatrix}$$

In[∘]:=
```
vout == mat.vin
```
Out[∘]=

True

Basis change

In[∘]:=
```
bs = Basis[S[1]]
```
Out[∘]=

$$\left\{ \left|0_{S_1}\right\rangle, \left|1_{S_1}\right\rangle \right\}$$

In[∘]:=
```
MatrixForm /@ Matrix[bs]
```
Out[∘]=

$$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

In[∘]:=
```
nbs = KetNormalize /@ ProperStates[S[1]]
```
Out[∘]=

$$\left\{ -\frac{\left|0_S\right\rangle}{\sqrt{2}} + \frac{\left|1_S\right\rangle}{\sqrt{2}}, \frac{\left|0_S\right\rangle}{\sqrt{2}} + \frac{\left|1_S\right\rangle}{\sqrt{2}} \right\}$$

In[∘]:=
```
new = Matrix[nbs];
MatrixForm /@ new
```
Out[∘]=

$$\left\{ \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \right\}$$

In[∘]:=
```
Matrix[S[1]] // MatrixForm
```
Out[∘]//MatrixForm=

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

In[∘]:=
```
MatrixIn[S[1], nbs] // MatrixForm
```
Out[∘]//MatrixForm=

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Problem : Given the matrix G of an operator $\hat{G}$ in the computational basis, how can you get the new matrix representation of the same operator $\hat{G}$ in a new basis.

Summary

Keywords

- Superposition principle, wave-particle duality
- Complementarity principle
- Basis, computational basis
- Quantum operations
- Measurement
- Matrix representation
- Basis change

Related Links

- Q3 Tutorial: The Postulates of Quantum Mechanics
- A Quantum Workbook (Springer, 2022), Chapter 1.