ULTIMATE PRECISION OF DIRECT TOMOGRAPHY OF WAVE FUNCTIONS

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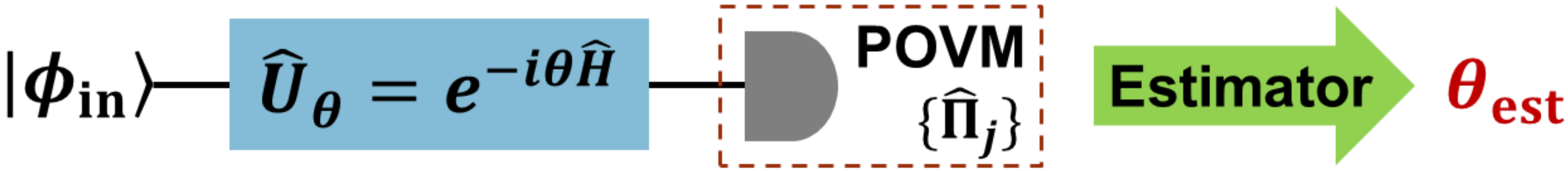
OUTLINE

1. Parameter estimation
2. Direct tomography of the wave function
3. Ultimate precision of dicrect tomography
4. Summary

1. PARAMETER ESTIMATION



The bound of unbiased estimate:

$$(\Delta\theta_{est})^2 := \langle(\theta_{est} - \theta)^2\rangle \geq \frac{1}{NF(\{\hat{\Pi}_j\})}$$

$$F(\{\hat{\Pi}_j\}) = \sum_j \frac{\left(\partial p_j/\partial\theta\right)^2}{p_j}$$ Fisher information

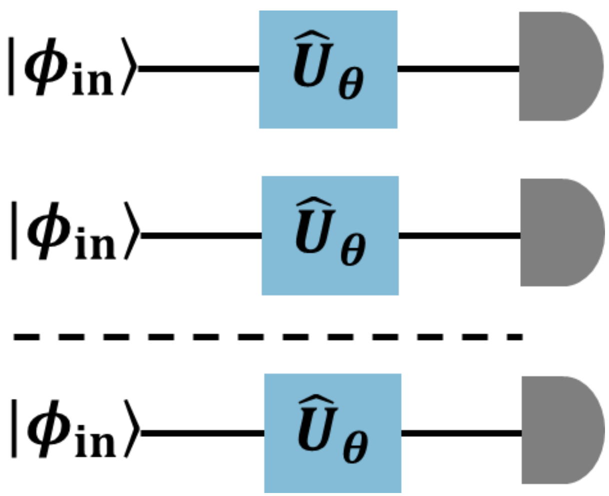
N: number of repetitions of experiment

$\{p_j\}$: probability distribution of $\{\hat{\Pi}_j\}$

Braunstein & Caves (PRL, 1994)

1. PARAMETER ESTIMATION

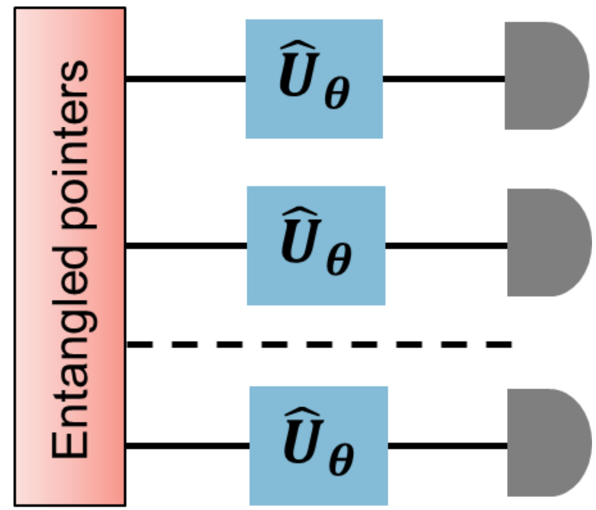
Quantum-enhanced metrology



$$(\Delta\theta_{est})^2 \sim \frac{1}{N}$$

Standard quantum limit

Pointer preparation



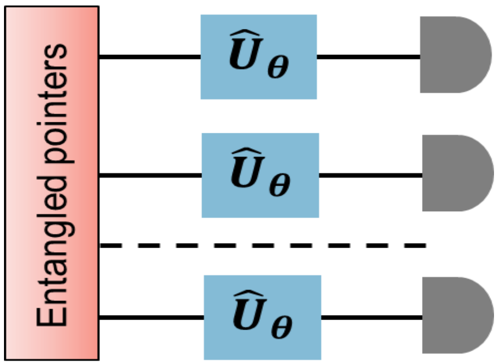
$$(\Delta\theta_{est})^2 \sim \frac{1}{N^2}$$

Heisenberg limit

Giovannetti *et.al.* (PRL, 2006)

1. PARAMETER ESTIMATION

Quantum-enhanced metrology



$$|0\rangle \to |0\rangle$$
$$|1\rangle \to e^{i\theta}|1\rangle$$

$$|\phi_{\text{in}}\rangle = \frac{|0\rangle^{\otimes N} + |1\rangle^{\otimes N}}{\sqrt{2}} \quad \to |\boldsymbol{\phi_{out}}\rangle = \frac{|\mathbf{0}\rangle^{\otimes N} + e^{iN\theta}|\mathbf{1}\rangle^{\otimes N}}{\sqrt{\mathbf{2}}}$$

Measure $\hat{\boldsymbol{\sigma}}_{x}$ on each probe (LOCC strategy)

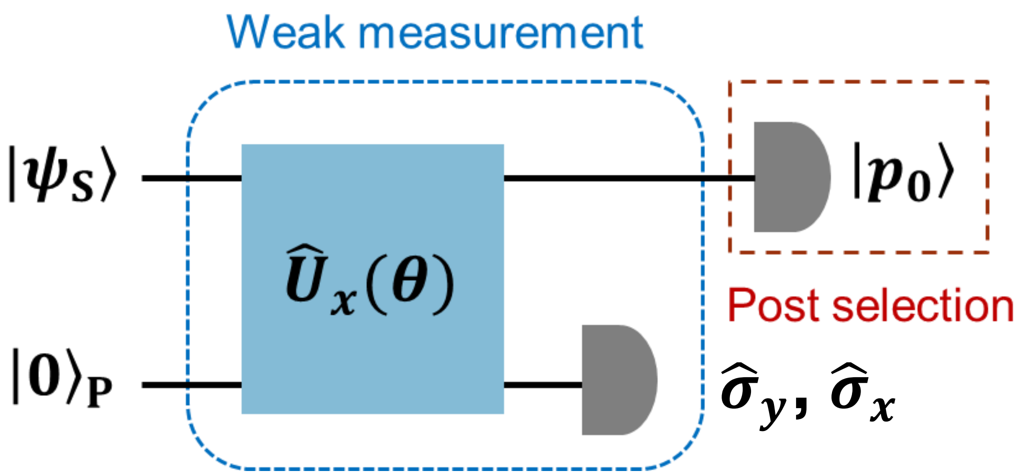
$$\langle \hat{\sigma}_x^{\otimes N} \rangle = \cos(N\theta)$$

$$\to \quad (\Delta\theta)^2 = \frac{1}{N^2}$$

Hwang Lee *et.al.* (J.mod.Opt, 2002)

2. DIRECT TOMOGRAPHY

Measure wave function directly based on weak measurement



$$|\psi_S\rangle = \sum_{x=1}^{d} \psi_x |x\rangle$$

$$\hat{U}_x = e^{-i\theta\,\hat{\Pi}_x \otimes \hat{\sigma}_y}$$

$$\approx \hat{I}_P - i\theta\,\hat{\Pi}_x \otimes \hat{\sigma}_y$$

$$|p_0\rangle = \frac{1}{\sqrt{d}} \sum_{x=1}^{d} |x\rangle$$

$$\langle \hat{\Pi}_x \rangle_w = \frac{\langle p_0 | x \rangle \langle x | \psi_S \rangle}{\langle p_0 | \psi_S \rangle} = k\psi_x$$

$$k = 1/\tilde{\psi}$$
$$\tilde{\psi} = \sum_x \psi_x$$

→ tomography of two-dimension state

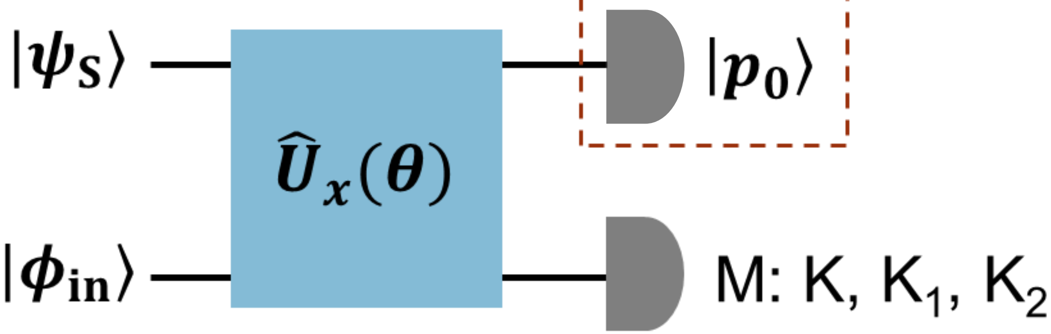
Ludeen et al. (Nature, 2011)

2. DIRECT TOMOGRAPHY

Measure wavefunction directly

No weak-coupling approximation

$|\psi_S\rangle = \sum_{x=1}^{d} \psi_x |x\rangle$



$\hat{U}_x = e^{-i\theta \hat{\Pi}_x \otimes \hat{K}/2}$

$|p_0\rangle = \frac{1}{\sqrt{d}} \sum |x\rangle$

$\hat{K}, \hat{K}_1, \hat{K}_2$: Pauli operators

$$\psi_x = \frac{d}{\tilde{\psi}\,\sin\theta}\left[\left(\left(1 - P_{K_1}\right)\tan\frac{\theta}{2} + P_{K_2} - \frac{1}{2}\right) + i\left(P_K - \frac{1}{2}\right)\right]$$

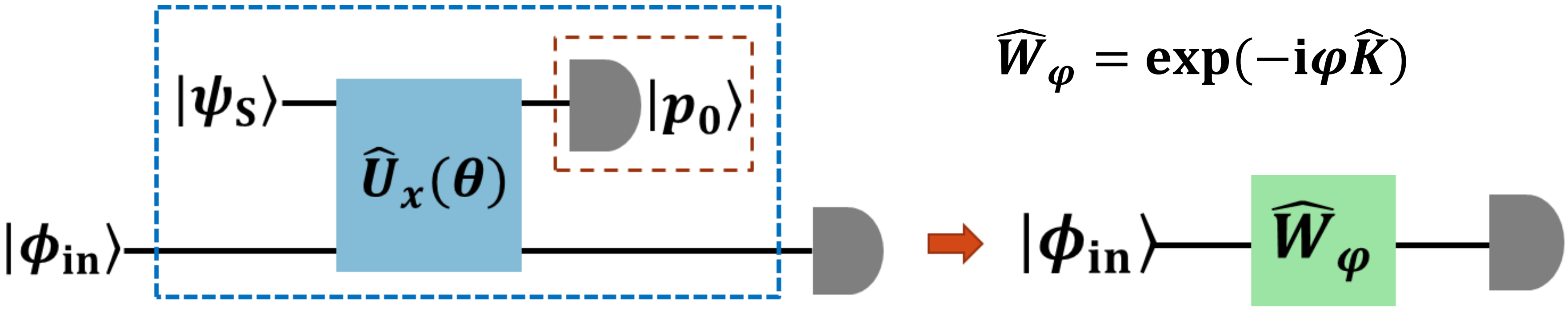
$\tilde{\psi} = \sum_x \psi_x$, $\quad P_M$: propability of measurement outcome 1

Vallone & Dequal (PRL, 2016)

2. DIRECT TOMOGRAPHY

as a complex-valued parameter estimation

$$\hat{W}_\varphi = \exp(-i\varphi\hat{K})$$



Final pointer state: $|\phi_f\rangle \propto \hat{W}_\varphi|\phi_{in}\rangle$

$\varphi \in \mathbb{C}$

Relation between ψ_x and φ:

$$\frac{\psi_x}{\tilde{\psi}} = \frac{\tan(\varphi/2)}{2\sin(\theta/2)[\cos(\theta/4) + \sin(\theta/4)\tan(\varphi/2)]}$$

X. H. T. Nguyen & M. S. Choi (arXiv:1911.07674v1)

2. DIRECT TOMOGRAPHY

Estimation of complex parameter

- The real part:

$$\begin{bmatrix}\langle\hat{K}_1\rangle_{\mathrm{f}}\\ \langle\hat{K}_2\rangle_{\mathrm{f}}\end{bmatrix} \propto \begin{bmatrix}\cos(\mathrm{Re}\varphi) & -\sin(\mathrm{Re}\varphi)\\ \sin(\mathrm{Re}\varphi) & \cos(\mathrm{Re}\varphi)\end{bmatrix}\begin{bmatrix}\langle\hat{K}_1\rangle_{\mathrm{in}}\\ \langle\hat{K}_2\rangle_{\mathrm{in}}\end{bmatrix}$$

- The imaginary part:

$$\langle\hat{K}\rangle_{\mathrm{f}} = \frac{\sinh(\mathrm{Im}\varphi) + \cosh(\mathrm{Im}\varphi)\langle\hat{K}\rangle_{\mathrm{in}}}{\cosh(\mathrm{Im}\varphi) + \sinh(\mathrm{Im}\varphi)\langle\hat{K}\rangle_{\mathrm{in}}}$$

If $\langle\hat{\boldsymbol{K}}\rangle_{\mathbf{in}} = \langle\hat{\boldsymbol{K}}_2\rangle_{\boldsymbol{in}} = \mathbf{0}$, two measurements are enough

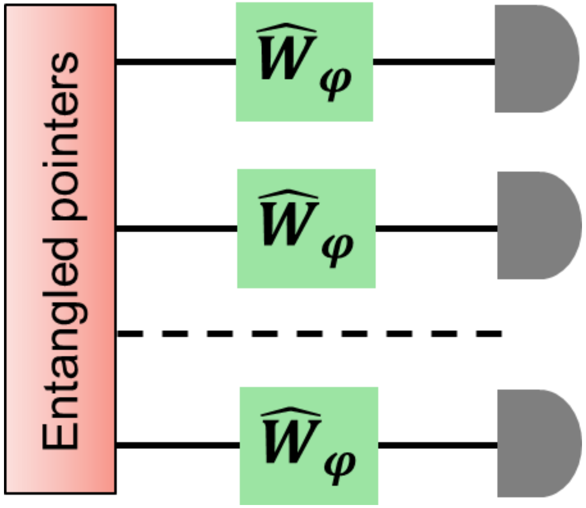
$$\langle\hat{\boldsymbol{K}}_1\rangle_{\mathbf{f}} = \frac{\cos(\mathbf{Re}\boldsymbol{\varphi})}{\cosh(\mathbf{Im}\boldsymbol{\varphi})}\langle\hat{\boldsymbol{K}}_1\rangle_{\mathbf{in}}, \qquad \langle\hat{\boldsymbol{K}}\rangle_{\mathbf{f}} = \frac{\mathbf{1}}{\mathbf{2}}\mathbf{tanh}(\mathbf{Im}\boldsymbol{\varphi})$$

X. H. T. Nguyen & M. S. Choi (arXiv:1911.07674v1)

2. DIRECT TOMOGRAPHY

Estimation of complex parameter

$\boldsymbol{\varphi} \in \mathbb{C}$



Cramer-Rao bound
for multi-parameter estimate

$$\boldsymbol{C}(\vec{\boldsymbol{X}}) \geq \boldsymbol{\mathcal{F}}^{-1}(\{\widehat{\boldsymbol{\Pi}}_j\})$$

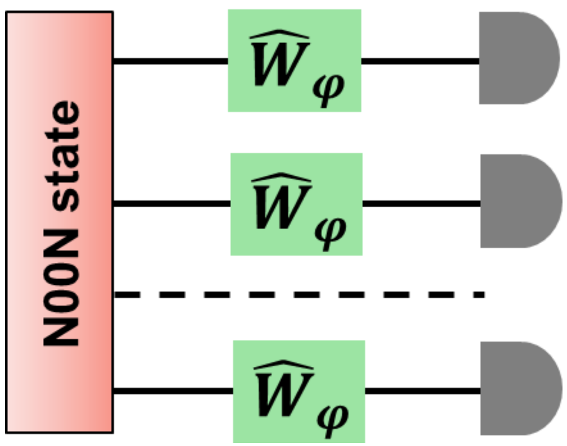
- Variance matrix: $C_{\mu\nu}(\vec{X}) = (\Delta X_\mu \Delta X_\nu)$
- Fisher information matrix:

$$\mathcal{F}_{\mu\nu} = \sum_j \frac{1}{p_j}\left(\frac{\partial p_j}{\partial X_\mu}\right)\left(\frac{\partial p_j}{\partial X_\nu}\right)$$

$$(\boldsymbol{\Delta\varphi_{\mathrm{est}}})^2 \sim \frac{\boldsymbol{1}}{\boldsymbol{N^2}}$$: **Heisenberg limit?**

3. ULTIMATE PRECISION OF DIRECT TOMOGRAPHY

Scheme 1a: Pointers are in NOON state $|\phi_{in}\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle^{\otimes N} + |1\rangle^{\otimes N}\right)$



Final pointer

$$|\phi_f\rangle_{\text{NOON}} = a|0\rangle^{\otimes N} + b e^{iN\text{Re}\varphi}|1\rangle^{\otimes N}$$

where $a^2 - b^2 = \tanh(N\text{Im}\varphi)$

$\widehat{W}_\varphi = \exp(-i\varphi\hat{\sigma}_z/2),\ \varphi \in \mathbb{C}$

- Measure $\hat{\sigma}_x$ to estimate $\text{Re}\varphi$
- Measure $\hat{\sigma}_z$ to estimate $\text{Im}\varphi$

X. H. T. Nguyen & M. S. Choi (arXiv:1911.07674v1)

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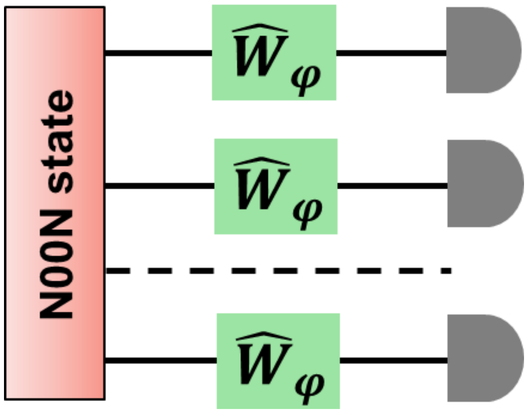


Heisenberg limit (as $\mathrm{Im}\varphi \to 0$)

$$\left(\mathcal{F}^{-1}\right)_{\mu\nu} \to \frac{1}{N^2}$$

$\widehat{W}_\varphi = \exp(-\mathrm{i}\varphi\hat{\sigma}_z/2),\ \varphi \in \mathbb{C}$

Can be proved by two measurements

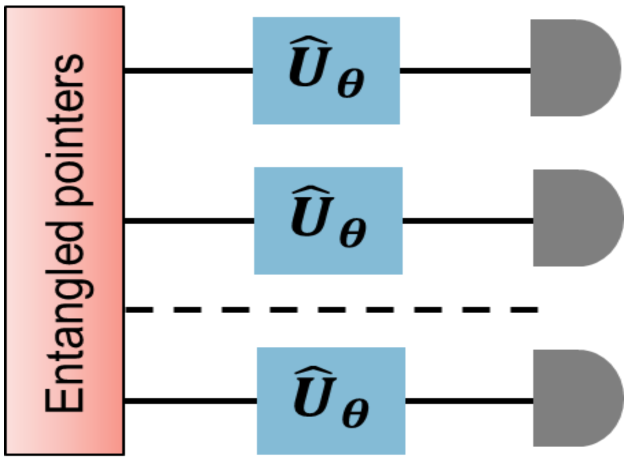
$$\langle\sigma_z^{\otimes N}\rangle = \tanh(N\mathrm{Im}\varphi), \qquad \langle\sigma_x^{\otimes N}\rangle = \frac{\cos(N\mathrm{Re}\varphi)}{\cosh(N\mathrm{Im}\varphi)}$$

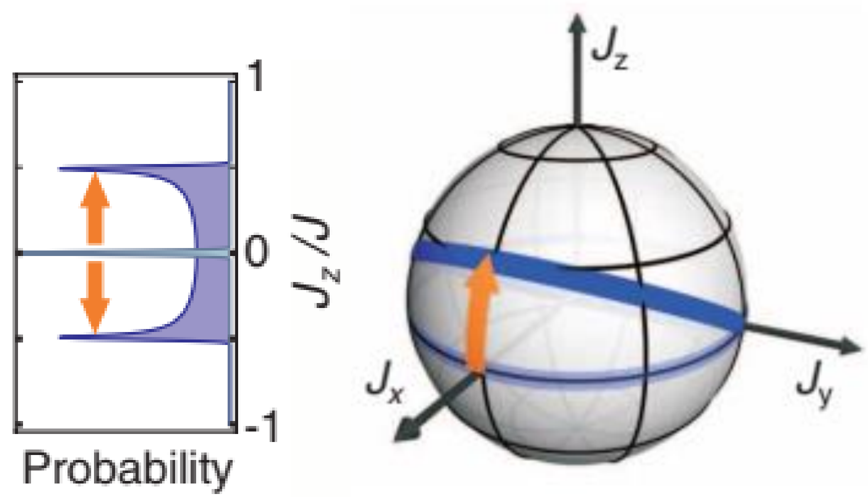
$$(\Delta\mathrm{Re}\varphi)^2 = (\Delta\mathrm{Im}\varphi)^2 = \frac{\cosh^2(N\mathrm{Im}\varphi)}{N^2} \to \frac{1}{N^2} \quad \text{as } N\mathrm{Im}\varphi \to 0$$

X. H. T. Nguyen & M. S. Choi (arXiv:1911.07674v1)

3. ULTIMATE PRECISION OF DIRECT TOMOGRAPHY

Scheme 1b: Pointers are in Dicke state $|\boldsymbol{\phi_{in}}\rangle = |\boldsymbol{j},\boldsymbol{0}\rangle$, $j = N/2$ integer





Measure $\hat{J}_z^2$ to attain Heisenberg limit in estimate rotation θ

Symmetric Dicke state

$$|\boldsymbol{D}_{N,N/2}\rangle := |\boldsymbol{j}_{=N/2},\boldsymbol{0}\rangle = \frac{\boldsymbol{j}!}{\sqrt{\boldsymbol{N}}}\sum_{\boldsymbol{P}} \hat{\boldsymbol{P}}|\underbrace{\boldsymbol{00}\cdots}_{j}\underbrace{\boldsymbol{11}\cdots}_{j}\rangle \rightarrow |\phi_{\text{out}}\rangle = e^{-i\theta \hat{J}_y}|j,0\rangle$$

Lücke *et al.* (Science 2011), Apellaniz *et al.* (New J. Phys. 2015)

3. ULTIMATE PRECISION OF DIRECT TOMOGRAPHY

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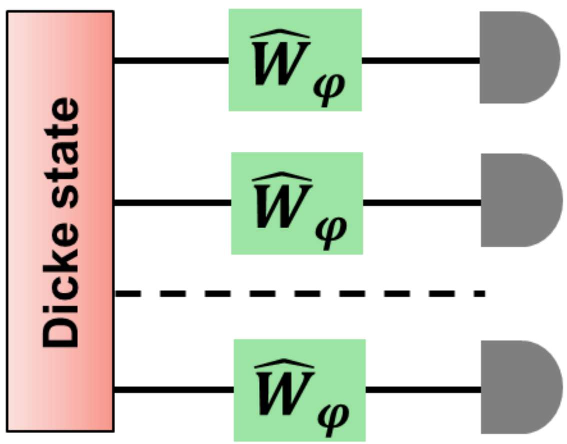
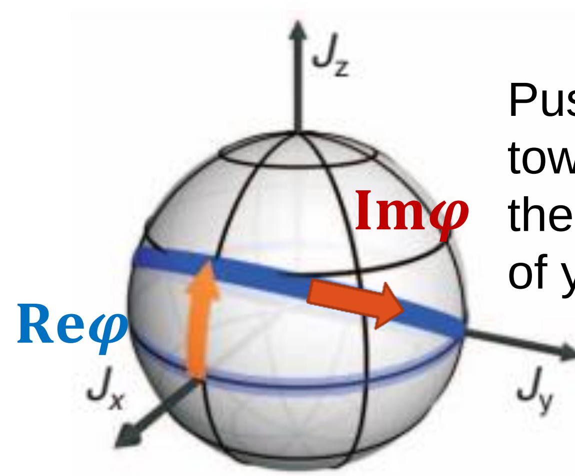


$\widehat{W}_\varphi = \exp(-i\varphi\hat{\sigma}_y/2)$, $\varphi \in \mathbb{C}$

Final pointer

$$|\phi_f\rangle_{\text{Dicke}} \propto e^{\text{Im}\varphi \hat{J}_y} e^{-i\text{Re}\varphi \hat{J}_y} |j, 0\rangle$$



Push toward the end of y-axis

Measure $\hat{J}_z^2 \rightarrow \text{Re}\varphi$

Measure $\hat{J}_y \rightarrow \text{Im}\varphi$

X. H. T. Nguyen & M. S. Choi (arXiv:1911.07674v1)

3. ULTIMATE PRECISION OF DIRECT TOMOGRAPHY

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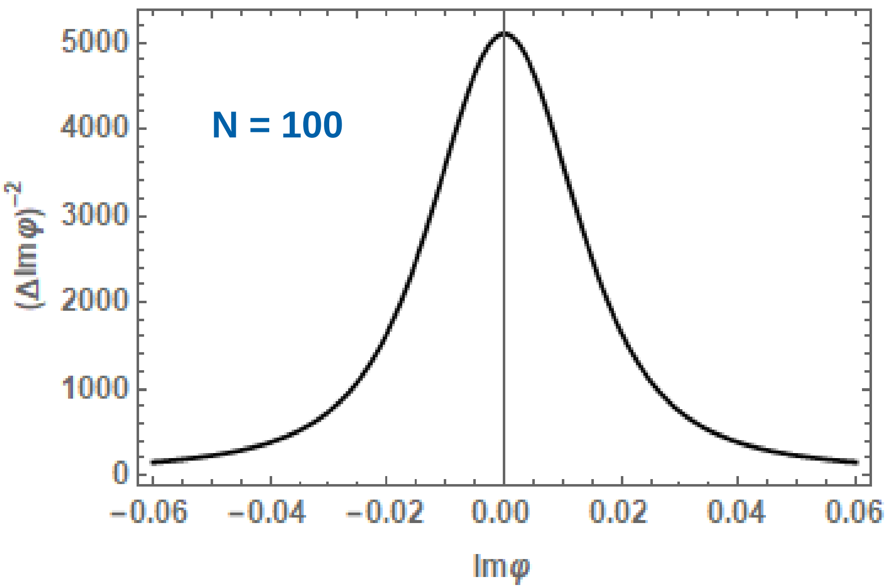
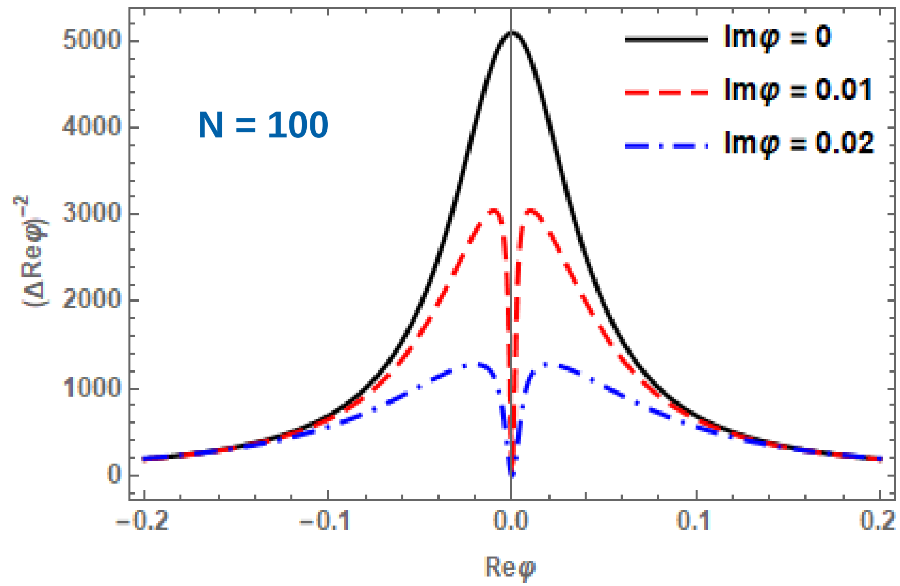




$$\mathrm{Im}\varphi \to 0: \; (\boldsymbol{\Delta}\mathbf{Im}\boldsymbol{\varphi})^2 \to \frac{\mathbf{2}}{\boldsymbol{N}(\boldsymbol{N}+\mathbf{2})}$$

$$\varphi \to 0: \; (\boldsymbol{\Delta}\mathbf{Re}\boldsymbol{\varphi})^2 \to \frac{\mathbf{2}}{\boldsymbol{N}(\boldsymbol{N}+\mathbf{2})}$$

Approaching Heisenberg limit

X. H. T. Nguyen & M. S. Choi (arXiv:1911.07674v1)

ULTIMATE PRECISION OF DIRECT TOMOGRAPHY

Pointers are entangled → Attainable Heisenberg limit in even complex-valued parameter estimate

N00N state pointers

$$\langle \sigma_z^{\otimes N} \rangle = \tanh(N \mathrm{Im}\varphi) , \qquad \langle \sigma_x^{\otimes N} \rangle = \frac{\cos(N \mathrm{Re}\varphi)}{\cosh(N \mathrm{Im}\varphi)}$$

Dicke state pointers

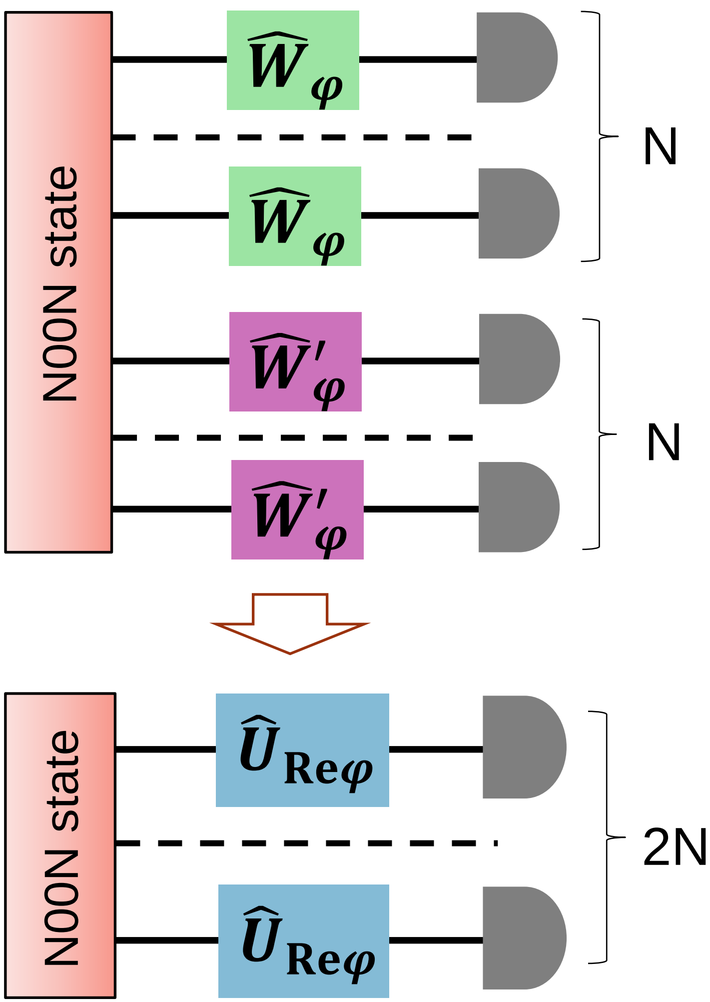
$$\langle \hat{J}_y \rangle = \frac{\mathrm{i} d_{10}^{(j)}(2\mathrm{i}\ \mathrm{Im}\varphi)}{d_{10}^{(j)}(2\mathrm{i}\ \mathrm{Im}\varphi)} \sqrt{j(j+1)}$$

$$\langle \hat{J}_z^2 \rangle = \frac{\mathrm{i} d_{10}^{(j)}(2\mathrm{i}\ \mathrm{Im}\varphi)}{d_{10}^{(j)}(2\mathrm{i}\ \mathrm{Im}\varphi)} \sqrt{j(j+1)} \left[\coth(2 \mathrm{Im}\varphi) \sin^2 \varphi + \frac{i}{2} \sin(2\varphi) \right]$$

Can we extract the only real part of φ?

ULTIMATE PRECISION OF DIRECT TOMOGRAPHY

Scheme 2: Time-inversal symmetry system ensemble



$$\widehat{W}_\varphi = \langle p_0 | \widehat{U}_x(\theta) | \psi_S \rangle$$

$$\widehat{W}'_\varphi = \langle p_0 | \widehat{U}_x(\theta) | \overline{\psi}_S \rangle$$

$$|\overline{\psi}_S\rangle := \widehat{T}|\psi_S\rangle = \sum_{x=1}^{d} \psi_x^* |x\rangle$$

$\widehat{T}$: Time-reversal symmetry operator

$$|\phi_f\rangle_{\mathrm{TRS}} = \frac{|0\rangle^{\otimes 2N} + e^{i2N\mathrm{Re}\varphi}|1\rangle^{\otimes 2N}}{\sqrt{2}}$$

$\rightarrow (\Delta\mathrm{Re}\varphi)^2 = \dfrac{1}{4N^2}$ Heisenberg limit

X. H. T. Nguyen & M. S. Choi (arXiv:1911.07674v1)

SUMMARY

- Direct tomography (no weak-coupling approximation) as a parameter estimation → optimal measurements
- Direct tomography attainable the Heisenberg limit:
  - Scheme 1: using maximally entangled pointers
  - Scheme 2: using time-reversal symmetry system ensemble