subject: universal two-qubit operations

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1 Algebraic preliminaries

A quantum computer is a controllable quantum system. Thus its primitive operations acting on a given Hilbert space are determined by its Hamiltonian. For instance, having a Hamiltonian

$$H = \alpha\sigma_1 + \beta\sigma_3 \tag{1}$$

the primitive operations are unitary operators generated by σ_1 and σ_3.

Despite of the widely accepted convenience of using `CNOT` and single unitary operators to generate the unitary group $\mathrm{U}(2^n)$, there are many quantum computing systems in which `CNOT` is not a primitive operator. For those systems it is often the case that the Hamiltonian of the system can be represented by sigma strings as in (1), so that primitive operations are generated by sigma strings.

For a concrete exposition, we may consider the following Hamiltonians for a single qubit and two-qubit system respectively:

$$H_1 = \alpha\sigma_1 + \beta\sigma_3, \quad H_2 = \alpha_1\sigma_{01} + \alpha_2\sigma_{10} + \beta_1\sigma_{03} + \beta_2\sigma_{30} + \gamma\sigma_{22}$$

with which primitive operations are generated by the following:

$$G_1 = \{\sigma_1, \sigma_3\}, \quad G_2 = \{\sigma_{01}, \sigma_{10}, \sigma_{03}, \sigma_{30}, \sigma_{22}\}. \tag{2}$$

Here,

$$\sigma_0 = -\frac{i}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \quad \sigma_1 = \frac{1}{2}\begin{pmatrix}0 & i\\ i & 0\end{pmatrix}, \quad \sigma_2 = \frac{1}{2}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}, \quad \sigma_3 = \frac{1}{2}\begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix}$$

and $\sigma_{ab} = 2i\sigma_a \otimes \sigma_b$.

In order to demonstrate G_1 and G_2 can generate $U(2)$ and $U(4)$ respectively, we employ the following decomposition of special unitary groups SU(2) and SU(4) [1]

$$\begin{aligned}
\mathrm{SU}(2) =& \exp(\theta_1\sigma_3)\exp(\theta_2\sigma_1)\exp(\theta_3\sigma_3)\\
\mathrm{SU}(4) =& K_2(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6)A_2(\beta_1,\beta_2,\beta_3)K_2(\gamma_1,\gamma_2,\gamma_3,\gamma_4,\gamma_5,\gamma_6)
\end{aligned}$$

where

$$\begin{aligned}
K_2(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6) =& \exp(\alpha_1\sigma_{03})\exp(\alpha_2\sigma_{01})\exp(\alpha_3\sigma_{03})\exp(\alpha_4\sigma_{30})\exp(\alpha_5\sigma_{10})\exp(\alpha_6\sigma_{30}),\\
A_2(\beta_1,\beta_2,\beta_3) =& \exp(\beta_1\sigma_{11})\exp(\beta_2\sigma_{22})\exp(\beta_3\sigma_{33}).
\end{aligned} \tag{3}$$

One can easily notice that SU(2) can be generated by G_1, but for SU(4) there exist non-primitive operators generated by σ_{11} and σ_{33}.

However, one can still implement those non-primitive operators by using G_2. To this enc, we consider the following:

$$\exp(\theta\sigma_A) = \cos\left(\frac{\theta}{2}\right)\mathbf{1} + 2\sin\left(\frac{\theta}{2}\right)\sigma_A, \quad A = ab, \quad a,b = 0,1,2,3.$$

Then for $[\sigma_B, \sigma_A] \neq 0$

$$\begin{aligned}\exp(\varphi\sigma_B)\exp(\theta\sigma_A)\exp(-\varphi\sigma_B) &= \cos\left(\frac{\theta}{2}\right)\mathbf{1} + 2\sin\left(\frac{\theta}{2}\right)\exp(\varphi\mathrm{ad}\,\sigma_B)\sigma_A \\ &= \cos\left(\frac{\theta}{2}\right)\mathbf{1} + 2\sin\left(\frac{\theta}{2}\right)\cos\varphi\sigma_A + \sin\left(\frac{\theta}{2}\right)\sin\varphi[\sigma_B, \sigma_A].\end{aligned}$$

In turn, one obtains

$$\exp\left(\frac{\pi}{2}\sigma_B\right)\exp(\theta\sigma_A)\exp\left(-\frac{\pi}{2}\sigma_B\right) = \cos\theta\mathbf{1} + \sin\theta[\sigma_B, \sigma_A] = \exp(\theta[\sigma_B, \sigma_A]).$$

which implies

$$\begin{aligned}\exp(\theta\sigma_{11}) &= \exp\left(-\frac{\pi}{2}\sigma_{30}\right)\exp\left(-\frac{\pi}{2}\sigma_{03}\right)\exp(\theta\sigma_{22})\exp\left(\frac{\pi}{2}\sigma_{03}\right)\exp\left(\frac{\pi}{2}\sigma_{30}\right) \\ \exp(\theta\sigma_{33}) &= \exp\left(\frac{\pi}{2}\sigma_{10}\right)\exp\left(\frac{\pi}{2}\sigma_{01}\right)\exp(\theta\sigma_{22})\exp\left(-\frac{\pi}{2}\sigma_{01}\right)\exp\left(-\frac{\pi}{2}\sigma_{10}\right).\end{aligned} \tag{4}$$

2 Computation project

Given decompositions in (3) one can attempt to find parameters to generate quantum logic gates `H`, `S`, `T` and `CNOT`. Also, one can replace σ_{11} and σ_{33} as in (4) to experimentally demonstrate that generators in (2) are sufficient to generate all elements in $\mathrm{U}(2)$ and $\mathrm{U}(4)$ up to the overall phase factor.

References

[1] N. Khaneja and S.J. Glaser, *Cartan decomposition of SU(2n) and control of spin systems*, *Chemical Physics* **267** (2001) 11.